The pagination in the text follows the original typescript, whose pagination differs somewhat from that of *Davis 1965.*

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Remarks before the Princeton bicentennial conference on problems in mathematics (*1946*)

Tarski has stressed in his lecture (and I think justly) the great importance of the concept of general recursiveness (or Turing's computability). It seems to me that this importance is largely due to the fact that with this concept one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion, i.e., one not depending on the formalism chosen.[1] In all other cases treated previously, such as demonstrability or definability, one has been able to define them only relative to a given language, and for each individual language it is clear that the one thus obtained is not the one looked for. For the concept of computability, however, although it is merely a special kind of demonstrability or decidability, the situation is different. By a kind of miracle it is not necessary to distinguish orders, and the diagonal procedure does not lead outside the defined notion. This, I think, should encourage one to expect the same thing to be possible also in other cases (such as demonstrability or definability). It is true that for these other cases there exist certain negative results, such as the incompleteness of every formalism or the paradox of Richard. But closer examination shows that these results do not make a

[1][*Footnote added in 1965*: To be more precise: a function of integers is computable in any formal system containing arithmetic if and only if it is computable in arithmetic, where a function f is called computable in S if there is in S a computable term representing f.]

definition of the absolute notions concerned impossible under all circumstances, but only exclude certain ways of defining them, or, at least, that certain very closely related concepts may be definable in an absolute sense.

Let us consider, e.g., the concept of demonstrability. It is well known that, in whichever way you make it precise by means of a formalism, the contemplation of this very formalism gives rise to new axioms which are exactly as evident and justified as those with which you started, and that this process of extension can be iterated into the transfinite. So there cannot exist any formalism which would embrace all these steps; but this does not exclude that all these steps (or at least all of them which give something new for the domain | of propositions in which you are interested) could 2
be described and collected together in some non-constructive way. In set theory, e.g., the successive extensions can most conveniently be represented by stronger and stronger axioms of infinity. It is certainly impossible to give a combinational and decidable characterization of what an axiom of infinity is; but there might exist, e.g., a characterization of the following sort: An axiom of infinity is a proposition which has a certain (decidable) formal structure and which in addition is true. Such a concept of demonstrability might have the required closure property, i.e., the following could be true: Any proof for a set-theoretic theorem in the next higher system above set theory (i.e., any proof involving the concept of truth which I just used) is replaceable by a proof from such an axiom of infinity. It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets.

Let me consider a second example where I can give somewhat more definite suggestions, namely the concept of definability (or, to be more exact, of mathematical definability). Here also you have, corresponding to the transfinite hierarchy of formal systems, a transfinite hierarchy of concepts of definability. Again it is not possible to collect together all these languages in one, as long as you have a finitistic concept of language, i.e., as long as you require that a language must have a finite number of primitive terms. But, if you drop this condition, it does become possible (at least as far as it is necessary for the purpose), namely, by means of a language which has as many primitive terms as you wish to consider steps in this hierarchy of languages, i.e., as many as there are ordinal numbers. The simplest way of doing it is to take the ordinals themselves as primitive terms. So one is led to the concept of definability in terms of ordinals, | i.e., 3
definability by expressions containing names of ordinal numbers and logical constants, including quantification referring to sets. This concept should, I think, be investigated. It can be proved that it has the required closure property: By introducing the notion of truth for this whole transfinite language, i.e., by going over to the next language, you will obtain no new

definable sets (although you will obtain new definable properties of sets).

The concept of constructible set I used in the consistency proof for the continuum hypothesis can be obtained in a very similar way, i.e., as a kind of definability in terms of ordinal numbers; but, comparing constructibility with the concept of definability just outlined, you will find that not all logical means of definition are admitted in the definition of constructible sets. Namely, quantification is admitted only with respect to constructible sets and not with respect to sets in general. This has the consequence that you can actually define sets, and even sets of integers, for which you cannot prove that they are constructible (although this can of course be consistently assumed). For this reason, I think constructibility cannot be considered as a satisfactory formulation of definability.

But now, coming back to the definition of definability I suggested, it might be objected that the introduction of all ordinals as primitive terms is too cheap a way out of the difficulty, and that the concept thus obtained completely fails to agree with the intuitive concept we set out to make precise, because there exist undenumerably many sets definable in this sense. There is certainly some justification in this objection. For it has some plausibility that all things conceivable by us are denumerable, even if you disregard the question of expressibility in some language. But, on the other hand, there is much to be said in favor of the concept under consideration; namely, above all it is clear that, if the concept of mathematical definability is to be itself mathematically definable, it must necessarily be so that all ordinal numbers are definable, because otherwise you could
4 define the first ordinal number not definable, and | would thus obtain a contradiction. I think this does not mean that a concept of definability satisfying the postulate of denumerability is impossible, but only that it would involve some extramathematical element concerning the psychology of the being who deals with mathematics.

But, irrespective of what the answer to this question may be, I would think that "definability in terms of ordinals", even if it is not an adequate formulation for "comprehensibility by our mind", is at least an adequate formulation in an absolute sense for a closely related property of sets, namely, the property of "being formed according to a law" as opposed to "being formed by a random choice of the elements". For, in the ordinals there is certainly no element of randomness, and hence neither in sets defined in terms of them. This is particularly clear if you consider von Neumann's definition of ordinals, because it is not based on any well-ordering relations of sets, which may very well involve some random element.

You may have noticed that, in both examples I gave, the concepts arrived at or envisaged were not absolute in the strictest sense, but only with respect to a certain system of things, namely the sets as conceived in axiomatic set theory; i.e., although there exist proofs and definitions not falling under these concepts, these definitions and proofs give, or are to give, nothing new within the domain of sets and of propositions expressible in terms of "set", "ϵ" and the logical constants. The question whether the two epistemological concepts considered, or any others, can be treated in a completely absolute way is of an entirely different nature.

In conclusion I would like to say that, irrespective of whether the concept of definability suggested in this lecture corresponds to certain intuitive notions, it has some intrinsic mathematical interest; in particular, there are two questions arising in connection with it: (1) Whether the sets definable in this sense satisfy the axioms of set theory. I think this question is to be answered in the affirmative, and so will lead to another, and probably simpler, proof for the consistency of the axiom of choice. (2) It follows from the axiom of replacement that the ordinals necessary to define all sets of integers which can at all be defined in this way will have an upper limit. I doubt that it will be possible to prove that this upper limit is ω_1, as in the case of the constructible sets.[2]

[2][*Footnote added on 26 June 1968*: I have recently been informed that this conjecture has been verified by Kenneth McAloon in a dissertation at the University of California at Berkeley: to be more precise, that Dr. McAloon, using Cohen's method, has proved the consistency (with the Zermelo–Fraenkel axioms of set theory) of the assumption that all sets are 'ordinal definable' and that $2^{\aleph_0}$ is much greater than $\aleph_1$.]