*Leibniz’s syncategorematic infinitesimals II:*

*their existence, their use and their role in*

*the justification of the Differential Calculus*

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**Abstract:**

In this paper we endeavour to give a historically accurate presentation of how Leibniz understood his infinitesimals, and how he justified their use. No one contests that Leibniz categorized his infinitesimals as fictions, but what this means has long been a matter of dispute. It has been alleged that notwithstanding his earlier attempts to present them as non-existent, his efforts to defend the calculus against the criticisms of Rolle and others at the turn of the century involved a different interpretation of “fiction”, where in order to explain the fruitfulness of methods involving infinitesimals he became committed to their existence as non-Archimedean elements of the continuum. Against this, we show that by 1676 Leibniz had already developed an interpretation from which he never wavered, according to which infinitesimals, like infinite wholes, cannot be regarded as existing because their concepts entail contradiction, even though they may be used as if they exist under certain specified conditions—a conception he later characterized as “syncategorematic”. Thus the question of existence must be distinguished from questions concerning the use and justification of infinitesimals: one cannot infer their existence from the success of methods based on them. By a detailed analysis of Leibniz’s arguments in his *De quadratura* of 1675-6, we show that Leibniz had already presented there two strategies for justifying infinitesimalist methods, one in which they stand for finite quantities that can be made as small as necessary in order for the error to be smaller than can be assigned, and thus zero; and another “direct” method in which the infinite and infinitely small are introduced by a fiction analogous to imaginary roots in algebra, and to points at infinity in projective geometry. We then show how in his mature papers the latter strategy, now articulated as based on the Law of Continuity, is presented (to critics of the calculus who do not accept Archimedean methods of justification) as being equally constitutive for the foundations of algebra and geometry, and also as being provably rigorous according to the accepted standards in keeping with the Archimedean axiom.

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Even though these entities [sc. lines and angles smaller than any assignable] are fictitious, geometry nevertheless exhibits real truths which can be expressed in other ways without them. But these fictitious entities are excellent abbreviations for expressions, and for this reason extremely useful. (*Numeri infiniti*, c. April 10, 1676; A VI 3, 499/LLC 89-91)

1. *Introduction: A simple dichotomy?*

The correct understanding of Leibniz’s infinitesimals has been the source of controversy ever since he introduced them in his first published papers in the 1680s. Interpretations have tended to fall on one side or another over the question of their existence: either there really are infinitesimals, that is, infinitely small actual parts of the continuum; or they are mere *façons de parler*, which do not designate any fixed entities.[[1]](#footnote-1) If the latter is true, it is thought, then infinitesimals must be dispensable, with methods using them replaceable by ones using only finite terms.[[2]](#footnote-2) By contraposition, it has seemed to many, beginning with Johann Bernoulli, the Marquis de l’Hôpital and Fontenelle, that if they are an indispensable tool in higher mathematics, then they must exist as components of the continuum.

The tendency to reduce the issue to a simple dichotomy—either for or against infinitesimals—has been fortified by the history of analysis over the last 140 years. Weierstrass had given a precise formulation of Cauchy’s concept of a limit, and had utilized this to give the ε-δ definition of the derivative of a function familiar from modern calculus textbooks, which does not involve differentials. To mathematicians suspicious of infinitesimals, this success seemed to make them redundant. Cantor in particular, as is well known, vehemently rejected infinitesimals, despite his advocacy of the actual infinite and creation of the theory of transfinite numbers. In this he was followed by Bertrand Russell, who claimed that Cantor and Weierstrass had banished infinitesimals from mathematics. This is still a view widely held by mathematicians and philosophers. But the picture changed substantially with the intervention of Abraham Robinson, who in 1966 (Robinson 1966) showed how to put infinitesimals on a mathematically sound foundation using his Nonstandard Analysis (NSA). This showed that one can construct a consistent theory of infinitesimals as quantities smaller than all finite quantities (thus violating the so-called Archimedean axiom to which Cantor had implicitly appealed in his alleged refutation.) Since then it has been realized not only that infinitesimals had enjoyed a continuous use in other branches of mathematics than analysis, but also that the theories of non-Archimedean infinitesimals proposed by Cantor’s contemporaries (Veronese, Stolz, Levi-Civita and others) could be built upon and improved.[[3]](#footnote-3) Such developments have led to calls for introductory calculus to be taught using real infinitesimals and elementary non-standard analysis. Moreover, they are often presented as a way to vindicate Leibniz’s point of view.[[4]](#footnote-4)

Leibniz himself, we argue, did his best to avoid subscribing to such a dichotomy. On the one hand, he firmly rejected as contradictory Bernoulli’s notion of infinitesimals as fixed elements of the continuum that are smaller than all finite quantities, arguing instead for a subtle view in which they have only a relational existence and should be considered as fictions. On the other hand, though, he held that their being fictions does not preclude their being used to express and generate truths, providing they are used in a way consistent with their fictional status. Accordingly, Leibniz was an enthusiastic advocate of infinitesimalist methods, stressing their ease of use and fecundity for making new discoveries in mathematics, analogous to the way in which algebra is superior to traditional geometry. In his *De quadratura arithmetica circuli ellipseos et hyperbolae* of 1675-76 (hereafter DQA) he gave a justification of quadrature (integration) from first principles (in a way that anticipated Riemannian integration) by appeal to the Archimedean Axiom applied to variable geometric quantities, and in later papers showed how to justify differentiation techniques by treating infinitesimals as fictional entities *incomparable* with finite quantities. He also developed a justification by appeal to an analogy with the way his contemporaries used imaginary roots and points at infinity to establish truths in algebra and in geometry. Once these justifications were established, infinitesimal methods could be pursued using fictional infinitesimals directly, without needing to provide a finitist proof for each instance of their use. Moreover, results could be obtained, for instance with transcendental functions, that could not be achieved using traditional geometrical methods depending on a double *reductio ad absurdum*. Consequently, Leibniz stressed that mathematicians could and should avail themselves of his methods using infinitesimals without first having to decide whether they exist as actual parts of the continuum, despite having his own very definite views on that question.[[5]](#footnote-5) Our first aim will be to clarify this view of infinitesimals by relying on neglected material and indicating how the late justification of the differential algorithm can be related to the arguments first provided by the DQA.

In this way we also aim to provide responses to criticisms on several fronts. It has been objected that although Leibniz characterized infinitesimals as non-existent fictions in DQA and other writings of the mid-1670s, it cannot be assumed that he continued to hold the same view of fictions after he had fully developed the differential calculus; and that in his mature work, Leibniz’s insistence on the fecundity of the calculus and the indispensability of infinitesimals showed a commitment to their existence as fictional elements of the continuum. This is the issue we tackle in section 2, where we show how Leibniz never deviated from his view that infinitesimals do not in fact exist. The question of existence, however, must be distinguished from that of usability. For even if they do not exist, fictions may still be used to obtain truths, so long as one can provide an accompanying demonstration.

A second line of criticism concerns the status of the DQA. It has been charged that despite the focus of commentators on Leibniz’s much lauded Proposition 6 of that treatise, infinitesimals are only introduced in a later proposition, that this is little more than a variation on the traditional method of indivisibles, that the treatise deals only with “well behaved curves”, as opposed to the more general ones treatable by the methods of the differential and integral calculus.[[6]](#footnote-6) In response, we show in section 3 how Leibniz’s method in the DQA builds on and improves upon the extant method of indivisibles, and then set about exploring some of the subtleties. In particular, we explain the significance of the facts that Proposition 7 of the treatise is *not* direct, and that the language of infinitesimals is not introduced before Prop. 8. It has also been claimed that the DQA is a work of juvenilia that Leibniz repudiated in his maturity, and whose methods he not longer valued, as evidenced by the (alleged) facts that he did not quote results from it or comment on in his later publications. In section 4 we provide a discussion of the many instances where Leibniz does in fact quote from, reference, and discuss the DQA in his publications and correspondence.

All of this allow us to provide what we think to be a new reading of the texts on the foundations of the calculus in the 1700s, in particular the famous *Cum Prodiisset* (section 5). We claim that the introduction of the Law of Continuity as a postulate does not amount to a change of views on infinitesimals, but to a second strategy as regards their use, which is fully compatible with the “syncategorematic” view. Moreover, we show that the justifications provided in this context can easily be couched in terms of the strategy sketched in the DQA, as Leibniz always claimed.

2. Fictions and the existence of infinitesimals

The problem of the composition of the continuum was of profound importance to Leibniz from his earliest writings till the end of his life. It is interesting to note that at the beginning of his career Leibniz advocated an interpretation of infinitesimals as actually existing entities in the continuum (see Arthur 2009). Thus in Leibniz’s writings of 1669 one can see some interesting points of correspondence with the theory of non-Archimedean infinitesimals of Edward Nelson’s “Internal Set Theory”. Similarly, in the *Theoria Motus Abstracti* that he sent to the Académie Royale des Sciences and the Royal Society of London in 1672, Leibniz advocated the existence in the continuum of actually infinitely many “unassignables”.[[7]](#footnote-7) These he conceived as infinitely small indivisibles lacking extension, but still having magnitude, in opposition to Galileo’s *parte non quanti*. He defined them as “smaller than can be expressed by a ratio to another sensible magnitude unless the ratio is infinite”, or as “rudiments … of lines and figures smaller than any that can be given” (A VI 2, 264-5/LLC 339-40). But by 1676 Leibniz had renounced all such construals of the infinitely small as actual parts of the continuum (whether indivisible or divisible) in favour of the view that they are *fictions*. Since some authors persist in holding that Leibniz’s fictionalist interpretation was a late development, prompted by the criticisms of Rolle and others in the 1690s, it will be worth reviewing the origins of this interpretation in the 1670s.[[8]](#footnote-8)

The fictionalist interpretation of the infinite can already be seen emerging in Leibniz’s analysis of quadratures and infinite series in the middle of his sojourn in Paris, as has been explained elsewhere. Thus in papers written in 1674 Leibniz uses his emerging proto-calculus methods to analyze the area under a hyperbola between one axis (as an asymptote) and a line parallel to it, calculating the area as an infinite series A = 1/1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + … . Using the same method, he is able to represent a finite area contained within A by the alternating series P = 1/1 – 1/2 + 1/3 – 1/4 + 1/5 – 1/6 + … . Subtracting, P – A = 1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + … = A. Thus the part of the area P – A is equal to the whole A, in contradiction to the axiom that the whole is greater than its (proper) part. Leibniz comments: “By this argument it is concluded that the infinite is not a whole, but only a fiction, since otherwise the part would be equal to the whole” (A VII 3, 468; October 1674). Here the infinite area is that between the hyperbola and its asymptote, and Leibniz argues that since taking it as a true whole leads to contradiction with the axiom that the whole is greater than its (proper) part, it should instead be regarded as a fiction. Similarly, he had earlier argued in some critical comments on Galileo’s *Discorsi* in 1672 that the part-whole axiom must be upheld even in the infinite. It follows that it is impossible to regard “all the numbers” and “all the square numbers” as true wholes, since then the latter would be a proper part of the former, and yet equal to it, yielding a contradiction. “Or”, he adds, “perhaps we should say that one ought not to say anything about the infinite except when there is a demonstration of it” (A VI 3, 168; LLC 9).

This remains Leibniz’s position into his maturity and both arguments are to be found, for example, in the correspondence with Bernoulli in 1698 (see section 4 below).[[9]](#footnote-9) That is, he held that the part-whole axiom is constitutive of quantity, so that the concept of an infinite quantity, such as an infinite number or an infinite whole, involves a contradiction. In fact, he even believed that the part-whole axiom is an analytic truth, demonstrable from the principle of identities and the definitions of whole and part.[[10]](#footnote-10) “The infinite”, Leibniz still claims in 1712, “whether continuous or discrete, is properly neither one, nor a whole, nor a quantum, and if we take it as such by a certain analogy, it is only, so to speak, a way of speaking; since, namely, there are more things than can be comprised by any number, yet we attribute a number to these things analogically, which we call infinite.”[[11]](#footnote-11) That is only half the story, however, as indicated in Leibniz’s above-quoted comments on the *Discorsi*. We can still use fictions such as infinite quantity in our reasoning, even if they cannot be supposed to exist without involving contradictions, so long as there is an accompanying demonstration. For fictions can be used to discover truths.

A short digression on Leibniz’s theory of knowledge will help clarify how this is possible. Unlike most modern epistemologists, Leibniz, considered many (if not most) of the concepts we ordinarily use to be subject to a “blind” or “symbolic” knowledge.[[12]](#footnote-12) This is a term of art for Leibniz. It means that even a “clear and distinct” concept—i. e one for which we can provide a nominal definition (allowing us to distinguish the entity in question—may harbour a hidden contradiction, which appears when analysing all of its constituents.[[13]](#footnote-13) Still, one can use such concepts for deriving truth. This is the case with the notion of a mathematical fiction applied to an infinite quantity. Although this concept contains a contradiction, other subsidiary concepts contained in it may permit the derivation of true entailments. For truths, according to Leibniz, consist in relations between concepts, so even a blind concept that virtually contains a contradiction may still (so long as we can “insulate” the derivation from the part containing the contradiction) be used to discover truths.[[14]](#footnote-14) Thus when Leibniz says that he understands the infinitely small to be a fiction, this is not a way of deflecting criticism by simply abjuring infinitesimals, as is sometimes assumed. It means that even though their concept may contain a contradiction, it can nevertheless be used to discover truths, provided a demonstration can (in principle) be given to show that its being used according to some definite rules will avoid contradiction.[[15]](#footnote-15) This strategy of using “fictions” is not limited to mathematics and was very widespread in Law, the discipline which Leibniz first learned as a student, where it took the form of the “*fictio juris*”.[[16]](#footnote-16) It also serves to introduce fictional entities in other domains of knowledge such as physics or philosophy. Interestingly enough, in both cases, Leibniz sometimes make the parallel with mathematics explicit and present the use of fictional infinitely small quantities not as an invention of his own, but as a common practice amongst mathematicians.[[17]](#footnote-17)

Here it should be noted that this already distinguishes Leibniz’s position from Robinson’s. For what Robinson demonstrated with his Non-Standard Analysis was precisely that one can introduce infinite numbers and infinitesimals *without contradiction*. The whole point of any “non-standard” approach is the building of a non-standard model in which what is said about entities such as “infinite numbers” or “infinitely small quantities” has to be literally and rigorously true. By contrast, Leibniz claims that what is said about infinitesimals is not true “à la rigueur” and entails a paradoxical way of speaking.[[18]](#footnote-18) A more accurate and fruitful comparison could be made with the use of “V” as the “set of all sets” in standard Set Theory. As is well known, there is no such thing as the “set of all sets”, on pain of contradiction (as shown by Russell with his famous paradox). Still set theorists working in standard ZFC have no problem introducing the *fiction* of such a model, on the basis of some classical caveat. Here is an example of such a caveat:

Informally, we call any collection of the form {x: φ(x)} a class. We allow φto have free variables other than x, which are thought of as parameters upon which the class depends. A *proper class* is a class which does not form a set (because it is “too big”). The Comprehension Axiom says that any subclass of a set is a set. We use boldface letters to denote classes. Two useful classes, which are proper by Theorems 5.2 and 7.4, are given by the following:

9.1 **DEFINITION.**

**V** = {x: x = x)

**ON** = {x: x is an ordinal}.

Formally, proper classes do not exist, and expressions involving them must be thought of as abbreviations for expressions not involving them. (Kunen 1980, 23-24)

This corresponds very well to the way Leibniz uses mathematical fictions as *compendia loquendi* and may also serve to recall that the use of such fictions is still common in contemporary mathematics satisfying all the standards of rigour.[[19]](#footnote-19)

Of course, if the infinite is not a whole, one might wonder about the legitimacy of calculating with it in mathematics, as Leibniz himself was doing in his manipulations of infinite series. For example, in extending the “Difference Principle” that he had articulated for finite series to apply also to infinite series, Leibniz had assumed a last term for converging infinite series, equal to 0. But he knew this needed justifying. Thus in one of the fragments in which he calculates the sum of the reciprocal triangular numbers, the *Theorema Arithmeticae Infinitorum* of Fall 1674, he acknowledges that the use of the principle “ought to be demonstrated to come out in the infinite” (A VII 3, 362), and sets about providing such a demonstration. He calculates an expression for the *y*th term of each series concerned, where “*y* signifies any number whatever”. By this means he is able to show that the sum of the reciprocal triangular numbers, 2{1 – 1/(*y* + 1)}, approaches 2 arbitrarily closely as *y* is taken arbitrarily large (A VII 3, 363).

By this means a fictional last or infinitieth term can be included in calculations involving the Difference Principle. In the example of the “the infinite space between Apollonius’s Hyperbola and its asymptote” and its corresponding infinite series A = 1/1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + …, , with last term 1/∞ or 0, Leibniz says we “must understand this 0, or nought—or rather, in this place, a quantity infinitely or unassignably small—to be greater or smaller according as we have assumed the last denominator of this infinite series of fractions, which is itself also infinite, smaller or greater” (A VI 3, 282/LLC 115).

Thus to say that a converging infinite series has a finite sum S does not involve adding actually infinitely many terms, since there is no last term. Rather, it means that the sum can be made sufficiently close to S by taking sufficiently many terms in a finite series with the same first term and law of the series. Leibniz explicitly says this in a fragment (“Infinite Numbers”) from the beginning of April, 1676:[[20]](#footnote-20)

Whenever it is said that a certain infinite series of numbers has a sum, I am of the opinion that all that is being said is that any finite series with the same rule has a sum, and that the error always diminishes as the series increases, so that it becomes as small as we would like. For numbers do not in themselves go absolutely to infinity, since then there would be a greatest number. (A VI 3, 503/LLC 98-99)

Thus an infinite series is infinite in the sense that no matter how many terms are taken, there are more; but it is not an infinite whole, a collection, or a set. And its sum is equal to a given number only in the sense that it can be made arbitrarily close to that number by taking sufficiently many terms in a finite series with the same law.

Underlying this justification for summing a converging infinite series there is a general principle that if the difference between two quantities can be made smaller than any pre-assigned quantity, then they are equal.[[21]](#footnote-21) This principle (which we call the *Principle of Unassignable Difference*, or PUD) may be rendered as follows:

If the difference between two quantities (or ratios of quantities) can be made smaller than any pre-assigned difference (by varying a co-dependent quantity), so that the error in equating them is smaller than any pre-assigned error, then the difference is null and the two quantities (or ratios of quantities) are equal.

The inclusion of “(or ratios of quantities)” is a nod to the way that Isaac Newton formulated the same principle in his unpublished *Treatise on Fluxions*, and later (in phoronomic guise) in his *Principia*.[[22]](#footnote-22) It is closely related to what Leibniz will later promote as his Law of Continuity, which he first published in the *Nouvelles de la république des lettres* in July 1687:

*When the difference between two instances in a given series, or in whatever is presupposed, can be diminished until it becomes smaller than any given quantity whatever, the corresponding difference in what is sought, or what results, must of necessity also be diminished or become less than any given quantity whatever*. (A VI 4, 371, 2032)

Leibniz continues to uphold the same conception of infinite series in his mature writings. Thus when his friend and correspondent Johann Bernoulli urges him to concede that an infinite series must have an infinite number of terms, and that there is therefore an *infinitesimus* term, so that “it and all the following terms” would be infinitely small (A III, 7, 874), Leibniz replies:

Let us suppose that in a line its 1/2, 1/4, 1/8, 1/16, 1/32, etc. parts are actually assigned, and that all the terms of the series actually exist. You infer from this that there also exists an infinitieth term. I, on the other hand, think that nothing follows from this other than that there actually exists any assignable finite fraction, however small you please. Similarly in motion, even though it passes through all the points, it still does not follow that there are two points infinitely near to one another, and even less does it follow that they are next to one another. And in fact I conceive points, not as elements of a line, but as limits or negations of further progress, or as endpoints of lines. (to Johann Bernoulli, 22 August/1 September 1698, GM III 536; A III 7, 885)[[23]](#footnote-23)

As this indicates, for Leibniz the infinitely small is likewise a fiction. The last or infinitieth term in a converging infinite series is not an infinitely small actual, but one may calculate with it as if it is on the understanding that it stands for the fact that one can substitute for it a finite term as small as you please. This licenses proceeding as if there are infinitely small entities, without being committed to their existence. As we shall discuss further below, this is the essence of the *Lemmas on Incomparables* that Leibniz had published in 1689 in the *Mémoires de Trévoux*, according to which “if someone does not want to employ *infinitely small* quantities, one can take quantities as small as one judges sufficient for them to be incomparable, so that they produce an error of no importance, indeed one smaller than a given error” (GM VI 168).

When Bernoulli persists in advocating actual infinitesimals, Leibniz repeats the position we have described above concerning the non-existence of infinite wholes, despite the multiplicity of their terms:

We can conceive an infinite series consisting merely of finite terms, or terms ordered in a decreasing geometric progression. I concede the infinite multiplicity of terms, but this multiplicity forms neither a number nor one whole. It means only that there are more terms than can be designated by a number; just as there is for instance a multiplicity or complex of all numbers; but this multiplicity is neither a number nor one whole. (To Johann Bernoulli, Feb. 21, 1699, GM III 575)

So we can see that the idea that Leibniz subscribed to a “Bernoullian continuum”, containing actual infinitesimals as its elements, is emphatically rejected by Leibniz himself, writing to the same Bernoulli.

Moreover, Leibniz is consistent in his denial of infinite wholes, infinitely small magnitudes, and infinitely great ones, maintaining this same position in all his writings from 1676 onwards. In the *New Essays* of 1704-05 he writes:

It would be a mistake to try to suppose an absolute space which is an infinite whole made up of parts. There is no such thing: it is a notion which implies a contradiction; and these infinite wholes, and their opposites the infinitesimals, have no place except in geometrical calculations, just like the imaginary roots in algebra. (*New Essays*, A VI 6, 158)

To Des Bosses he writes in 1706:

Speaking philosophically, I maintain that there are no more infinitely small magnitudes than there are infinitely large ones, that is, no more infinitesimals than infinituples. For I hold both to be fictions of the mind through an abbreviated way of speaking, adapted to calculation, as imaginary roots in algebra are too. Meanwhile I have demonstrated that these expressions are very useful for abbreviating thought and thus for discovery, and cannot lead to error, since it suffices to substitute for the infinitely small something as small as one wishes, so that the error is smaller than any given, whence it follows that there can be no error. (GP II 305/LDB 32/3)

and to Luigi Grandi in September 1713:

Moreover, my opinion, very often expounded, is that infinitely small as well as infinite quantities are indeed fictions, although useful for reasoning compendiously and at the same time safely. And it suffices that they are understood to be truly as small as is necessary in order for the error to be smaller than any given; from which it is shown that there is no error. (GM IV 218)

Now, a word on the “syncategorematic infinite”. On various occasions in the 1700s Leibniz noted that his view of the infinite as a term not denoting a whole or collection, but as nevertheless being meaningful and usable when understood correctly, has a precedent in scholastic philosophy. For certain medieval philosophers, including Gregory of Rimini and William of Ockham, held that the infinite should be understood “syncategorematically”, that is, as deriving its meaning and its legitimation from the context in which it is used.[[24]](#footnote-24) Thus earlier in the section of the *New Essays* quoted from above, Leibniz wrote:

It is perfectly correct to say that there is an infinity of things, i.e. that there are always more of them than can be specified. But it is easy to demonstrate that there is no infinite number, nor any infinite line or other infinite quantity, if these are taken to be genuine wholes. The Scholastics were taking this view, or should have been doing so, when they allowed a syncategorematic infinite, as they called it, and not a categorematic one. (*New Essays*, A VI 6, 157)

The syncategorematic infinite was often identified with the Aristotelian potential infinite. But unlike Aristotle, Leibniz held that there are actually infinitely many things—not simply that no matter how many are specified there *could be* more, but (as he says here) that there *are* always more, even though there is no infinite number. (This is the sense in which the Scholastics “should have been” interpreting the syncategorematic infinite.) For example, Leibniz insisted that since every body is actually divided by motions within it into further bodies that are themselves similarly divided without bound, bodies “are actually infinite, that is to say, more bodies can be found than there are unities in any given number whatever” (A VI 4, 1393; LLC 235).[[25]](#footnote-25) That is, their multiplicity “surpasses every finite number”.[[26]](#footnote-26)

Similarly, when Pierre Varignon asked Leibniz for clarification of his views on the infinitely small in 1702, Leibniz replied (February 2, 1702) that it had not been his intention “to assert that there are in nature infinitely small lines in all rigour, or compared with ours, nor that there are lines infinitely greater than ours”. This is why, in order to avoid these subtleties, he had proposed that “it suffices to explain the infinite by the incomparable, that is to say, to conceive quantities incomparably greater or smaller than ours” (GM IV 91/ L 543). Nevertheless, he adds, this does not undermine calculations with the infinite,

for there always remains a syncategorematic infinite, as they call it in the Schools, and it remains true that 2 is as much as 1/1 + ½ + ¼ + 1/8 + 1/16 + 1/32 + …, which is an infinite series in which all the fractions whose numerators are 1 and whose denominators are in geometric progression of powers of 2 are comprised at once, even though only ordinary numbers are used and no infinitely small fraction, or one whose denominator is an infinite number, ever occurs in it. (GM IV 93/L 544)

This is entirely consistent with the account of converging infinite series Leibniz had sketched in 1676, and then explained again to Bernoulli in 1698. There is an actual infinity of terms, understood syncategorematically: no matter how many terms one takes, there are always more. There is no last, infinitely small term in such a series, nor is there any infinitely small fraction; although one may calculate as if there is a last term of zero on the understanding that by taking sufficiently many terms in a finite series with the same law, the error can be made less than any pre-assigned error. But there are no categorematically infinite entities, such as actually infinite magnitudes or quantities, or actually infinitely small elements in the continuum. Thus for Leibniz there is an actual infinite in multitude. This is the infinite understood syncategorematically and distributively: there are infinitely many *x* in the sense that no matter how many are designated, there are more. In this sense, for instance, there are infinitely many odd numbers, infinitely many even numbers, infinitely many primes, etc. But there is no actually infinite collection or number of them.

In sum: Leibniz first characterizes the infinite and the infinitely small as fictions because they violate the axiom that the whole is greater than its (proper) part. In deliberate opposition to Galileo, he insists that this axiom applies to the infinite as well as to the finite; for him, indeed, it is constitutive of the very idea of quantity. If one wants to treat infinites as genuine mathematical entities, they must be quantities, and thus subject to the part-whole axiom; but then to treat them as wholes results in contradiction. One can nonetheless use infinities and infinitely small quantities in a calculation provided one can furnish a way of doing correct demonstrations with them; that is, as long as one can identify conditions under which their use will not lead to error.

That is the issue of justification, to which we now turn. As a first pass, let us first examine the implications of the masterwork on quadrature that Leibniz produced in Paris in 1675-76, the *De quadratura arithmetica circuli ellipseos et hyperbolae*,or DQA for short, where Leibniz mentions his interpretation in terms of fictions*.* For it has been charged by critics that this treatise had no lasting significance for Leibniz’s understanding of the calculus, and that his treatment of infinitesimals and the infinite has scant relevance to his later defences of the calculus. To counter such charges, we will first give a presentation of some of the underappreciated aspects of the treatise, and then proceed to an examination of its later career in Leibniz’s thinking.

3. The DQA and the “direct method”

Since the rediscovery of the DQA in the 1990s, several accounts have been given of its main theorem, Proposition 6, and its significance. It was published in a critical edition with commentary by Eberhard Knobloch in (Leibniz 1993), who drew attention to the central role played by Prop. 6. Leibniz describes that proposition in his *Index notabiliorum* as a “very thorny” [*spinosissima*] one,

in which it is demonstrated in fastidious detail how certain step-spaces, and likewise certain polygonal spaces, can be increased continuously, to the point where they differ from each other or from curves by a quantity smaller than any given, which is something that is most often [simply] assumed by other authors. Although one can skip over it at first reading, it serves to lay the foundations for the whole Method of Indivisibles in the soundest possible way. (Leibniz 1993, 24)

As has been shown in articles by Knobloch (2002), Arthur (2008, 2013), Levey (2008) and Rabouin (2011, 2015), the nub of the proof is an exploitation of the Archimedean property to prove that quantities whose difference can be reduced to a quantity smaller than any given quantity are equal. Knobloch regards this as a form of the Principle of Continuity; following Arthur (2008), we have called it the PUD, as explained above. Here we will limit ourselves to a general presentation, referring the reader to those descriptions for more details and emphasizing, by contrast, some aspects of the treatise that may have been hidden by the overemphasis on Prop. 6.

The basic technique used by Leibniz is the “transformation” (*transmutatio*) of a given curve into another one (which he calls elsewhere the *Quadratrix*). One first considers the lengths A1T, A2T intercepted on the ordinate axis (horizontal in the figure) by tangents at points 1C, 2C, etc. on the curve and reports them as abscissas 1T1D,2T2D, etc., for the new curve. One then reports in the same way the intersections of the chords passing through the points 1C, 2C, etc. into points on the ordinates of the Quadratrix : 1N, 1P, 2N, 2P, etc. This gives birth to a step space (*spatium gradiformis*) with each of the triangles A1C,2C, etc., being a half of each of the corresponding rectangles 1B1N1P2B, etc., since they have the same width and height.



Leibniz then has to demonstrate: first, that the areas under the two curves (not taken from the origin) can be estimated, respectively, by the polygon A1C2C 3C…nCA and by the step-space 1B1N1P2N2P…nNnPnB; second, that the relationship given term by term between triangles and rectangles holds for these two areas. At the time, the two techniques available for doing so were the traditional proof by exhaustion and the modern proof by indivisibles. The first method relied on inscribed and circumscribed polygons and concluded by a double *reductio ad absurdum* that the area could not be greater or smaller than the intended result without contradiction. The second worked with infinite collections and concluded by transferring some ratios holding for the finite case to the infinite case.

Both entailed well known limitations: the first method presupposed the construction of inscribed and circumscribed polygons to the curve; it also presupposed that one knew in advance the result that was to be obtained (and, because of the use of *reductio*, it had no heuristic or explicative power); the second one presupposed that an area can be identified with an infinite collection (typically “all the lines” under the curves), and that some ratio remains when passing from the finite to the infinite, a fact that remained in need of a rigorous justification;[[27]](#footnote-27) finally, even when people understood “all the lines” as a convenient way of talking about small rectangles under the curve, they usually presupposed that their widths were equal.[[28]](#footnote-28)

Leibniz’s proof is an original mixture of (and improvement on) both techniques. It is “in the manner of the Ancients”[[29]](#footnote-29) in the sense that it does not appeal to infinite collections (or infinitely small quantities, or “indivisibles”, etc.). But it shares with the method of indivisibles the advantage that one does not need a method for constructing a polygon under the curve,[[30]](#footnote-30) and that one can proceed directly by identifying the area under the *Quadratrix* with a collection of rectangles. To obtain this identification without inscribed and circumscribed figures, Leibniz uses rectangles passing through the curve at arbitrary points. Another originality is the systematic use of upper bounds for the estimation of the error, which in particular enables one to have unequal decompositions of the area under the *Quadratrix*.[[31]](#footnote-31)

Leibniz then proceeds to show that the usual methods of indivisibles are special cases of this general construction and emphasizes the fact that he has provided in this way a “rigorous” foundation for them.[[32]](#footnote-32) Here by “rigorous” Leibniz clearly means the way in which he showed the method of indivisibles (usually used without any justification) could be rephrased, as was usual at the time, “in the manner of the Ancients”. However, it should be emphasized that his method is *direct* and does not rely on a *reductio*. It should also be emphasized that it does not rely on “infinitely small” quantities either, even in the language of description. It is based, like the proof of the Ancients, on finite quantities, and the idea is that one can translate the usual proofs using “indivisibles” into ones involving only these finite quantities. The fundamental insight is to work directly on the “difference” (or “error”) between the approximating quantity and the area under the curve, taken as a variable quantity, which may be rendered as small as one wishes.[[33]](#footnote-33) This corresponds well to a kind of “reductionist” strategy to which Leibniz will continue to appeal late in his career:

For in place of the infinite or the infinitely small one takes quantities as great and as small as is necessary in order for the error to be less than the error given; which differs from the style of Archimedes only in the expressions, which are more direct in our method and better adapted to the art of discovery (to Pinsson, 29 August 1701, GM IV 96)

Although the focus in previous studies on Prop. 6 was justified, it also had the unfortunate consequence of concealing other very important aspects of this treatise, the first one being that Prop. 7 is *not* direct and that the language of infinitesimals is not introduced before Prop. 8. Let us dwell a little bit upon these two features.

Prop. 6 gives the general process on which the quadrature of the given curve relies, but it does not give the actual quadrature. Indeed, there is something missing in the proof: nothing ensures, in the given framework, that the relationship between each triangle under the first curve and the corresponding rectangle of the step-space will “hold in the limit”, as we would say nowadays (although the advantage of the “direct” method is to give a heuristic justification that this is the relationship we ought to demonstrate properly). To prove this fact, Leibniz relies on a *reductio*. He first assumes a difference Z between twice the *Trilineum* (T) and the *Quadrilineum* (Q). Then by reducing the distance between the points taken on the curve, he shows that the difference between the *Trilineum* and the Polygon P can be made smaller than one fourth of Z and the same for the difference between the *Quadrilineum* and the *Spatium Gradiformis*. But since this *Spatium Gradiformis* G is twice the area contained by the Polygon P (this is a *finite* sum), we finally get the following inequalities:

T – P < Z/4, hence 2P – 2T < 2Z/4

Q – G < Z/4 and G = 2P, hence Q – 2P < Z/4

From this, it follows that Z = Q – 2T < |Q – 2P| + |2P – 2T|[[34]](#footnote-34) = Z/4 + 2Z/4 = 3Z/4 < Z. *Quod est absurdum*.

The fact that Leibniz uses a *reductio* here is very striking and should not be underestimated.[[35]](#footnote-35) In particular, in order to render the difference between the two quantities smaller than 3Z/4, it should have sufficed to conclude directly by using the same reasoning as in Prop. 6. But this is not what Leibniz did, which indicates that to produce a direct proof was not a central issue here. Indeed, it is easy to see on the preceding reasoning that each direct proof in terms of unassignable differences is equivalent to a *reductio* concluding from the fact that a quantity is rendered smaller than itself—and vice versa.[[36]](#footnote-36) As we shall see, however, this kind of conversion is not immediate for other examples of “direct” proofs provided by Leibniz. We shall not enter here into an explanation of the choice of an indirect proof,[[37]](#footnote-37) but emphasize that it gives a specific meaning to the famous scholium inserted just after this proposition:

For my part I confess that there is no way that I know of up till now by which even a single quadrature can be perfectly demonstrated without an inference *ad absurdum*. Indeed, I have reasons for doubting that this would be possible through natural means without assuming fictitious quantities, namely, infinite and infinitely small ones; but of all inferences *ad absurdum* I believe none to be simpler and more natural, and more proper for a direct demonstration, than that which not only simply shows that the difference between two quantities is nothing, so that they are then equal (whereas otherwise it is usually proved by a double *reductio* that one is neither greater nor smaller than the other), but which also uses only one middle term, namely either inscribed or circumscribed, rather than both together; and so brings it about that we have clearer comprehensions of these matters. (Leibniz 1993, 35)

First, one should keep in mind that Leibniz did not use the “fiction” of “infinitely small” quantities either in Prop. 6 or in Prop. 7. So the horizon of the reference to a direct proof in the first sentence cannot be—at least, cannot solely be—the method used in these propositions (this will be confirmed by a study of Prop. 8). Second, one should notice that Leibniz praises here a certain method *amongst* all the *reductio* proofs (and not as an alternative to them). In sum, we have a *first* dictionary which allows the translating of descriptions in terms of indivisibles into ones in terms of finite quantities, and of *reductios* into (pseudo)-direct proofs, in propositions 6 and 7. But this first dictionary is not the one relying on fictions yet, and is not direct in this sense. Moreover, such a method had already been sketched by some authors such as Pascal.[[38]](#footnote-38) Rather than dwelling on existing methods, Leibniz (rightly) emphasized other aspects of his results: that he does not have to resort to inscribed and circumscribed polygons (and, accordingly, not to a double *reductio* either), that he has provided a general proof,[[39]](#footnote-39) and that he can rely on quadrature with rectangles and with triangles.[[40]](#footnote-40)

All of this is crucial for a proper reading of Prop. 8, the one in which the fiction of “infinitely small” quantities will be used for the first time.[[41]](#footnote-41) The purpose of proposition 8 is to generalize the result to the case in which one begins the quadrature at the origin of the curve. But why is there a particular problem here? The answer is simple: in this case, the geometric figures produced in the sub-divisions degenerate (*degenerare* is the verb used by Leibniz). The triangle under the first curve degenerates into a sector (which Leibniz calls a “segment” here) and the rectangle of the step space becomes an “orthogonal *trilineum*”.[[42]](#footnote-42) This is a crucial remark because in this case, there is, at first sight, no way one can apply the relationship between the basic triangle under the first curve and the corresponding rectangle of the step-space. Still, Leibniz maintains that the proof could consist “in one word” (*uno verbo confici potest*) since the relations used in proposition 7 were considered between arbitrary small segments and hence also holds in the infinitely small case (*Quae proporitione 7 demonstravimus generalia sunt, et locum habent, utcunque parvae sint rectae*). They hold in particular when A, 1C and 1B coincide, that is to say, when their distance is infinitely small (*ac proinde sint infinite parvae, sive etsi puncta coincidat*).

This kind of reasoning was already used by Leibniz as early as 1674 in a series of texts to which we shall return later, since they provide a first form of what would later be called the “Law of Continuity”.[[43]](#footnote-43) The context was the elaboration of a general formula for conic sections (one of Leibniz’s favourite examples when mentioning his law of continuity later) and the fact that it implies the consideration of some parameters as either going to infinity in some cases, or as vanishing in others.[[44]](#footnote-44) Thus, Leibniz remarks that in algebra one often needs some equation to hold even in degenerate cases, typically when points come to coincidence. The simplest example is a segment AB and a point C taken at random on the same straight line. If we want the equation C = AB ± BC to be general, “In this case of the coincidence of B and C we must conceive the line BC as infinitely small, so that the equation does not contradict the equality between AC and AB”.[[45]](#footnote-45)

The important point to notice in the first “proof” of Prop. 8 is the following: in this case, the proof is “direct”, and in fact immediate, in a different sense than in Prop. 6. By the introduction of the “fiction” of infinitely small quantities, we have indeed the degenerate case entering into the ordinary one, even if the geometrical shape of the objects under consideration is not preserved[[46]](#footnote-46). This directly contradicts a common view according to which Leibniz would have developed this kind of argument, based on continuity, as an alternative to the methods presented in the *Quadratura*. In fact, the introduction of “fiction” corresponds precisely to this argument (and not to proposition 6). This is made explicit in the Scholium to Prop. 23:

What we have said up to this point about infinities and infinitely small quantities will appear obscure to certain people, as does everything new—although we have said nothing that cannot be easily understood by each of them after a little reflection: indeed, whoever has understood it will recognize its fecundity. It does not matter whether there are such quantities in nature, for it suffices that they be introduced by a fiction, since they provide abbreviations of speaking and thinking, and thereby of discovery as well as of demonstration, so that *it is not always necessary to use inscribed or circumscribed figures and to infer* ad absurdum*, and to show that the error is smaller than any assignable*. Nevertheless it is evident that the latter can easily be done by means of the things we said in Props. 6, 7, and 8. (Leibniz 1993, 69; 2016, 128; our emphasis)

Although we cannot presume at that stage that the Law of Continuity acts in the same way here as in later texts, it is crucial to keep in mind that the DQA already presents two strategies of proof, one direct and one indirect, the one based on fictions being the first one. Moreover, Leibniz then proceeds to show that this direct proof is equivalent to the other one because it can be translated into it.

Leibniz therefore supposes a difference Z between the two figures and sets about showing that it leads to a contradiction. But he cannot use the relationship between the triangles and the rectangles directly, so here is how he proceeds: first he takes the rectangle A1B1C2T smaller than 1/4Z and concludes that its constituents, the small “segment” A1CA and the small *trilineum* A1B1DA are also smaller than 1/4Z. Then he considers these two figures as parts of bigger figures in which one part is not degenerate. Typically the small segment A1CA is the difference between a greater “segment” A3C 2CA and the sector 1C A3C 2C1C (a non-degenerate figure). By the same reasoning, the small *trilineum* is the difference between the large *trilineum* A3B3D2B2DA (the figure we are interested in) and the *quadrilinum* 1D1B3B3D2D1D (a non-degenerate figure). Since the intended relation holds between the non-degenerate figures, he can then conclude that the difference between the large *trilineum* and twice the small segment (seen as the first difference above) is smaller than 3/4Z.

Notice that in this case, the “translation” between the two proofs does not amount to simply taking, instead of infinitely small quantities, finite quantities which approach them more and more closely, because this would not solve our first problem (they don’t have the same shape as the non-degenerate figures). The reasoning is hence slightly more complicated and necessitates the introduction of a second type of differences (on both sides of the inequality).

Here we have a clear example of a translation between a “direct” proof relying on a continuity argument together with the fiction of infinitely small quantities, and a *reductio* proof in the manner of the Ancients. The feature we would like to emphasize is that this meaning of “direct” is different from the one relating the *reductio* in Prop. 7 to the direct proof in Prop. 6. This is a subtle issue since Prop. 8 proceeds by translating the direct proof into the method of Prop. 7, which itself can be rendered in terms of the direct construction of Prop. 6. This means that, at the end of the day, all of these methods rely on the *PUD*. But, this being said, they don’t proceed in the same way, in particular when it comes to their use of fictions. Only the third one relies constitutively on fictions and only this one concludes by using the continuity argument.

Now Prop. 8 is not the only proposition that has been neglected by scholars dealing with infinitesimals in DQA—which is surprising, considering the fact that this is where the concept is first introduced—also neglected is Prop. 11 This is a no less surprising fact, considering that this is where Leibniz proposes a clarification of his views on the infinite.

Let us recall the content of this proposition. Prop. 11 is intended to show the following (somewhat paradoxical) result: from any curvilinear figure, no matter how small, one can extract a part, which is twice the magnitude of a figure *with infinite length*.[[47]](#footnote-47)

In order to show this result, Leibniz starts again from the given curve 1C2C 3C…nC, but, instead of its origin A, considers an arbitrary point μ, where he draws a tangent to the curve μλ. He then draws a perpendicular to this tangent as an axis of coordinates, chooses a point A on this straight line where it meets the curve again[[48]](#footnote-48) and considers a perpendicular to this perpendicular as a second axis (this line being, by construction, parallel to μλ). By recourse to the previous construction, Leibniz can then perform exactly the same reasoning as in Prop. 6, with the only difference that the *Quadratrix* now has an asymptote μλ. Relying on his previous results, he can show that the area comprised between the *Quadratrix* (starting at 2D) and the asymptote — a figure which is infinite in length — is twice the portion of the area under the curve delimitated by the points 2C2Bμ3C2C.

Now the conceptual difficulty is of course to establish a finite ratio between a space infinite in length and a finite area. Here Leibniz comments that he knows no other way (*non aliter fieri potest*) than to introduce a point (μ) at infinitely small distance from the axis. In this case, indeed, the straight line (μ)λ will still be infinite in length in the sense than it can be made greater than any given quantity (*major qualibet assignabili)*, yet will be bounded (*terminata*). Accordingly the line “(μ)λ will not be an asymptote to the curve Dδ, but will cut it in a point λ, this point being distant by an infinite interval”. This kind of proof will allow the introduction of a kind of “arithmetic of the infinite” in which one explains the ratios between finite and infinite quantities (Prop. 20, then used in 21).[[49]](#footnote-49)

Following this proof, Leibniz comments on some paradoxical results occurring with the development of the “method of indivisibles”, and in particular on the famous construction by Torricelli of an “infinitely long solid” with a finite volume.[[50]](#footnote-50) According to him, this kind of result no longer seems mysterious as soon as one realizes that it is based on the fiction of a line (or a surface) at the same time *terminata* and *infinita*. We find here again the same strategy as the one he had devised for infinite series: the addition of a fictitious *terminatio* allows one to calculate with the infinite. It should be noticed that another way of describing the introduction of this *terminatio* would be the addition of a point at infinity — again one of Leibniz’s favourite examples when mentioning his Law of Continuity.[[51]](#footnote-51) *Caeteris paribus*, it resembles what is done in modern analysis by recourse to “compactification”: one transforms a space which cannot be covered by finite means into one which can by adding a point at infinity, and this allows one to calculate more easily—for example, some integrals.

It is here that Leibniz then comments about the difference between “indivisibles” and “infinitely small” quantities, in parallel with *interminatum* and *infinitum*. The crucial issue, as was already noticed by several authors at the time, is that of homogeneity, an indivisible and a non-terminated object being related in a heterogeneous manner to their whole and parts respectively. This is why, Leibniz explains, a rectangle with an infinitely small width and an infinite length can be equal to a finite rectangle, as happens when one performs the quadrature of the hyperbola. This leads to the following characterization in a variant Scholium to Prop. 11:

I call unbounded that in which no ultimate point can be assumed, at least on one side. But I call infinite a quantity, whether bounded or unbounded, whenever we conceive it as greater than any quantity that is assignable by us, that is to say, numerically expressible. To determine whether nature suffers such quantities is the business of the Metaphysician; for the Geometer, it suffices to demonstrate what results from supposing them. (Leibniz 1993, 133; 2016, 60)

We find here another very important feature of Leibniz’s strategy when dealing with infinite and infinitely small quantities: nowhere does he express his conviction that they are contradictory notions, although this was the way—as we have shown—by which he was led to his own conception of infinite wholes as fictions. Everything is done so that the surface language of the mathematician remains *neutral*, leaving the philosophers to entangle the difficult question of the *Labyrinthum continui*. Since mathematicians that Leibniz held in high esteem, such as Galileo or Gregory of St Vincent, did not share his views about the existence of non-standard quantities, it was of crucial importance to show that this question does not belong to mathematics proper because the mathematician can develop a language which can be interpreted in both ways.[[52]](#footnote-52) This is the strategy which Leibniz will follow in all of his public declarations from then on, and it is all the more surprising that some commentators maintain that one can infer from the surface language to the existence of entities populating the mathematical continuum.[[53]](#footnote-53)

Although there are many other very interesting propositions in the DQA, Props. 7 and 11 will be sufficient to help us clarify the connections between the syncategorematic view and the conduct of proofs. The syncategorematic view is a thesis about the infinite. It says that whenever we seem to make reference to an infinite quantity in mathematics, be it an infinite number, an infinitely long distance, an infinite bounded line or an infinitely small quantity, what we are doing in fact is pointing to some relations between finite quantities. As we have seen, Leibniz calls a quantity infinite, “whether it is bounded or unbounded, whenever we conceive it as greater than any quantity that is assignable by us, that is to say, numerically expressible” (Leibniz 1993, 68; 2016, 60). Taken as such, this view entails the possibility of paraphrasing any use of the infinite through sentences in which it does not occur (as was already the case in Ancient Geometry).

Now, it should be stressed that this does not tell us anything about the way to conduct the proofs.[[54]](#footnote-54) As the previous developments should make clear, this conception is compatible with (at least) two different kinds of proofs : proof “in the manner of the Ancients” in which one could replace the difference which can be taken *quantumlibet parva* by an “infinitely small” error;[[55]](#footnote-55) and a direct proof relying on continuity argument such as the “one sentence proof” at the beginning of Prop. 8. It is crucial to note that in this second case, the complete justification is *not* given, according to Leibniz, by the paraphrase in and of itself, because it is clear that those who don’t accept “infinitely small” quantities will not be convinced. The reason is profound: even if we translate the “one sentence proof” into a longer proof involving finite quantities that can be taken as small as we wish, the difficulty will not be solved since it relies on the fact that the shape of the objects under consideration degenerates. If we want to translate the direct proof into a “rigorous” one, we need to do more than just rephrase it without recourse to infinite quantities.

4. The posterity of the DQA

But what is the status of the DQA in the long term? Some critics have downplayed its significance for Leibniz’s later defence of the calculus, even while acknowledging its importance for understanding the genesis of Leibniz’s views. Indeed the treatise does not contain any Differential Calculus, and although it indicates a general strategy that could be useful in justifying the Leibnizian algorithm, the move toward this justification is not immediate.[[56]](#footnote-56) Moreover, the DQA appears in many aspects as a piece of juvenilia that Leibniz overestimated for political reasons (Leibniz wrote it for submission to the *Académie Royale* in the hope of securing a place in that illustrious assembly).[[57]](#footnote-57) It has been charged that it deals only with “well behaved curves”,[[58]](#footnote-58) and that it relies on a main result, which, from the point of view of the differential and integral calculus, is still limited.[[59]](#footnote-59) Finally, it is alleged that if this treatise had the significance that some commentators have given it, then it is striking that Leibniz did not rely on it when justifying the calculus in later periods. “What about the fact,” Victor Blåsjö objects, “that he didn’t submit it, nor publish it later, nor reproduce its key results in his extensive subsequent correspondence on the foundations of the calculus?” (Blåsjö 2017, 136).[[60]](#footnote-60) In the present section, we shall answer this question and more generally provide a reply to these various objections.

Beginning with the easiest argument, it is a historical fact that the last claim is false, as is more generally the idea that Leibniz gave up the DQA because it was obsolete and superseded by the techniques of the Differential Calculus. This fact has been known since the end of the 19th Century when Gerhardt published a *Compendium* of the DQA, which Leibniz prepared for publication and in which he had no trouble employing the differential algorithm. The two techniques are henceforth certainly not exclusive of one another. Thanks to philological tools, we now know that the *Compendium* was prepared at the beginning of the 1690s at the earliest.

Moreover, Leibniz certainly sought to publish the treatise itself. In an exchange of letters in 1682 he discusses the project of publishing the DQAin Holland. Ferguson found a publisher: Rieuwerts (Spinoza's publisher), but Leibniz finally decided that the DQA should be inserted in a more general discourse on *Ars Inveniendi*— and, as usual, he did not find the time to complete this project. More interestingly, though, ten years later Leibniz reopened the possibility of publishing the DQA. In a letter to Henri Basnage de Bauval of late August 1692, he wrote:

A certain Mons. Rieuwerts, a bookseller in Amsterdam who has published some writings of Spinoza—to whom the late M. Ferguson sent me—was disposed at one time to publish certain geometrical speculations that I had. But the distractions that I then had did not permit me to lay it out in full, and I contented myself by giving certain abstracts in the *Actes* of Leipzig. Perhaps Mons. Rieuwerts would again take charge of it, since it appears more plausible and more accessible to the majority of readers. One could add a preamble containing some curious particulars on what Mons. Descartes invented or took from elsewhere. (A II 2, 559)

This plan to relaunch publication was perhaps stimulated by Leibniz’s exchanges with Bodenhausen. For when Leibniz initiated Bodenhausen into the new algorithm (which he intended to publish as an appendix to the *Dynamica*, mimicking Newton’s strategy in his *Principia*), Bodenhausen raised a few *dubia*, to which Leibniz answered methodically.[[61]](#footnote-61) After a few exchanges, as he came to understand the calculus better, Bodenhausen signalled to Leibniz that it would be very useful to have at one’s disposal a gentle introduction to shut the mouths of the Euclidean “Pied Pipers” (*Rattenfänger*), who were hostile to the new method (A III 4, 564). Leibniz responded positively to the demand and sent as an appendix to his letter from October 26, 1690, a presentation of the calculus for those who were trained in the “manner of the Ancients”.[[62]](#footnote-62) And what did he provide on this occasion? A presentation of Prop. 6 of the DQA accompanied by a translation into the differential calculus corresponding to Prop. 8.[[63]](#footnote-63) To be sure, all of these results are at the time superseded by the many researches in which Leibniz had been engaged since then, and he does warn his correspondent that the results from the DQA are almost immediate with the new calculus.[[64]](#footnote-64) But precisely, it is all the more striking that when coming to a translation of this calculus into the language of the Ancients, the only example he has to provide in 1690 is still prop. 6 of the DQA. This confirms something that he already said when presenting the proof of Prop. 6 in detail: this kind of proof is boring and painful, satisfying only those who are scrupulous. However, one should perform such proofs as the occasion demands to confirm that the general strategy is well grounded.

This is precisely the strategy Leibniz adopts in 1695 in his exchange with Johann Christoph Sturm, who had found difficulty in understanding his quadrature. Leibniz responds:

Concerning my Quadrature, I am pleased to remove the difficulty that the distinguished gentleman has encountered. But since he seems to desire a demonstration, I attach here some foundations from which he will easily be able to resolve it, for there is no time to explain the whole thing now at length. I have, indeed, several ways of demonstrating the same thing, but this one seems most elegant. (A II 3, 102)[[65]](#footnote-65)

What Leibniz gives immediately following this is an outline of the main argument of the DQA, stretching over three pages (A II 3, 102-104). His First Lemma is Prop. 1 of the DQA, that showing the equivalence of the triangle construction with that of rectangles twice their size (Leibniz 1993, 24-25; see Rabouin 2015, 15-16). On this basis he constructs the transformed curve, his *Quadratrix*, so that the area under the first curve (decomposed into triangles) is transformed into an area under the second curve that is decomposed into rectangles twice their size. He then gives an epitome of the method of the DQA encapsulated in Props. 6 and 7, which allows him to calculate the quadrature. “From this theorem,” he writes,

there follows almost at once the quadrature of all the paraboloids and hyperboloids which Fermat, Wallis and others have squared by sums of numbers, by assuming a kind of induction. … And from this theorem I also found the absolute quadrature—hitherto unknown—of that segment of a cycloid which is cut off if you draw from the vertex two straight lines to a point on the curve, on which a parallel to the base drawn through the centre of the generating circle meets the curve. (A II 3, 103)

This describes Proposition 14 of the DQA (Leibniz 1993, 45; 2016, 66). He also reports his result in Prop. 32, that “the circle is to the circumscribed square, that is, the arc of the quadrant is to the diameter, as 1/1 – 1/3 + 1/5 – 1/7 + 1/9 – 1/11 etc. is to unity” (1993, 79; A II 3, 104). Here Leibniz adds that when he was in Paris, Prestet, “author of the *Elements of a Universal Mathematics*,[[66]](#footnote-66) had persuaded himself that the cycloid is an arc of this circle—which is impossible, as I easily showed him” on the basis of the foregoing arguments.[[67]](#footnote-67)

In sum, when Leibniz is pressed to explain his method of quadrature to someone having difficulty with it in 1695, the “most elegant” way he can conceive of demonstrating it is not in terms of the more powerful methods he has developed since his youth, but by reference to the very presentation in the DQAwhich those methods had, according to Jesseph and Blåsjö, rendered inadequate and obsolete. So it is certainly not the case that in the 1690s, flush with the success of his calculus, Leibniz regarded the DQA as containing nothing of interest to colleagues familiar with the new methods, or that he never saw fit to reproduce its key results.

Moreover, contrary to Blåsjö’s claim that he never quoted any of its results in foundational discussions, we have not only the example of the exchange with Bodenhausen of 1690, but also what Leibniz wrote to Johann Bernoulli in 1698. In a letter dated July 29, he returns to Proposition 22 of the DQA in connection with the notorious case mentioned above of Torricelli’s famous construction of an “infinitely long solid” with a finite volume, and how the air of paradox can be dispersed by recourse to the fiction of a line (or a surface) that is at the same time *terminata* and *infinita*. The context is an objection raised by De Volder “against the infinitesimal calculus”, which Leibniz credits Bernoulli with having “correctly solved”. It concerns a hyperbola whose equation is *y = a3/x2*, and the fact that if *x* is supposed infinitely small, then *y* is an infinity of second degree. Leibniz comments:

I myself once formulated a cognate objection to this very one in the Scholium of Proposition 22 of the unpublished treatise which I composed in France on my Arithmetical Quadrature, shortly after its discovery. There the objection appeared to apply not only to our calculus, but also with equal right to the geometry already accepted up to that time.

… [There follows a lengthy description of prop. 18 of the treatise, taking up about 40 lines with accompanying diagram (GM III 522-524; A II 3, 855-857).] …

But between ourselves I would also add this, that I also wrote in the said unpublished manuscript that it is possible to doubt whether there could be infinitely long straight lines that were nevertheless also in fact bounded. I wrote moreover that it suffices for the calculus that they be taken as fictions [*fingantur*], like imaginary roots in algebra. For it is always the case that what is concluded by means of the infinite and infinitely small can be evinced by a *reductio ad absurdum* by my method of incomparables (the Lemmas for which I gave in the *Acta*). So you also shouldn’t wonder that I doubt whether there is an infinitely small thing, or an infinitely great one bounded on both sides. For even though I concede that there is no portion of matter that is not actually cut, one does not on that account come to uncuttable elements, or minimum portions, nor indeed to infinitely small things, but only to ones perpetually smaller, and yet ordinary; similarly in increasing one comes to perpetually greater ones. (GM III 522-524; A III, 8, 855-858)

Here the connection between of the language of fictions from the 1690s and that of the DQA is made fully explicit. Indeed this last document is also crucial, since this is the place where Leibniz first publicly divulged to Bernoulli his idea that infinitesimals are imaginary entities that can be introduced by a kind of fiction. This is the very passage to which he will later refer when talking to Varignon of his “*fictions utiles*”:[[68]](#footnote-68)

For the rest, I wrote some years ago to Mons. Bernoulli of Groningen that the infinite and infinitely small could be taken as fictions, similarly to imaginary roots; without which a wrong would be done to our calculus, these fictions being useful and founded in realities. (14 April 1702, GM IV, 93-94)

Moreover, it is the beginning of a very long exchange in which Leibniz makes it clear that he does not share Bernoulli’s view on the continuum.[[69]](#footnote-69)

This is not the place to enter into a detailed commentary of all of these fascinating pieces. Our first aim is to emphasize that they exist: there are in fact many documents in which Leibniz refers to the DQA, most of the time very explicitly, as the place to go to find a justification for the use of infinitesimals. These documents have been so far completely neglected, if not ignored.

5. Leibniz’s mature justifications of the calculus

An important issue, often neglected by commentators, and which should be apparent from the previous section, is that Leibniz modulates his justifications according to the context in which such a justification is requested. This remark is in fact of great importance for assessing, by contrast, the texts usually mentioned in discussions of the justification of the differential calculus. Indeed most of these justifications, dating from around 1700, occur in a specific debate, that of the crisis occurring in the *Académie Royale* at the turn of the Century, after several attacks by mathematicians “of the old style” such as Gouye, La Hire, Gallois and Rolle.[[70]](#footnote-70) When he finally intervenes in the debate between Rolle and Varignon in 1701, Leibniz does not present a translation into the Archimedean style, it is true, and he seems to be content to allude to the possibility of such a translation. But there may be a very good reason for that, which is rarely mentioned. This is that for “Cartesians” such as Rolle, as was already the case for critics like Clüver, the reference to Archimedes was of little value, if any.[[71]](#footnote-71) Recall that Descartes in fact considered the Archimedean Geometry to be a kind of Metaphysics (of Geometry), the value of which could only be heuristic.[[72]](#footnote-72) Recall also that, although he was perfectly able to use indivisibilist methods, he rejected them in geometry, tackling questions such as the quadrature of the cycloid only in his correspondence (AT II, 135-136). Recall finally that he thought of the *reductio* as the lowest form of reasoning in mathematics.[[73]](#footnote-73) This may give an idea as to why a Cartesian could hardly be “convinced” by demonstrations “in the manner of the Ancients“.

This will also help us in identifying a specific kind of justification occurring in these debates. Indeed, Leibniz emphasizes in this context a different argument in favour of infinitesimals. It does not amount to defending their use directly, but to objecting to the supporters of Cartesian techniques or the like that they use similar foundations in “ordinary” algebra. Accordingly, if this kind of foundation is not reliable in the Differential Calculus, it should not be reliable in their own practice either.

The underlying idea itself is not new and goes back, as we have noticed, to the Parisian period. It appears for the first time in the *Méthode de l’universalité* (1674), where Leibniz explains that infinitely small and infinitely large quantities are hidden in ordinary algebra as long as one wishes certain formulae to be general.[[74]](#footnote-74) This forms the core of a text which Leibniz sent to Varignon for publication in the spring of 1702[[75]](#footnote-75) and which was significantly entitled: “*Justification du calcul des infinitesimales par celuy de l’algebre ordinaire*” (GM IV 104–106).

In it, Leibniz takes the example of two lines AX and EY meeting at C, draws the perpendiculars EA = *e* and XY = *y* and joins A to X (with AX = *x* and AC = *c*). Hence he obtains two similar triangles CAE and CXY, or, in equation. Now, he considers what happens when EY moves toward A, assuming that the two triangles are not equal and that the angle in C is not 45 degrees. It is clear that the equation holds whatever the size of CEA might be. But what happens when C and E finally coincide with A? In this case, *c* and *e* vanish and *x* – *c* = *x*. If we take the vanishing of *c* and *e* to amount to “absolute zeros”, this leads us to an absurd statement: will be equal to (which Leibniz takes to signify that they are equal or of ratio 1). This means that *c* and *e* should not be considered as absolute nothings, but as relative ones, nothing *in comparison with* *x* and *y*.[[76]](#footnote-76) However, they need to keep a certain ratio, if we want the equation to be general, “and so they are treated as infinitesimals, exactly as are the elements which our differential calculus recognizes in the ordinates of curves for momentary increments and decrements. Thus we find in the calculations of ordinary algebra traces of the transcendent differential calculus and the same peculiarities about which some scholars have scruples.” (GM IV 104)[[77]](#footnote-77)



We find here the same kind of continuity argument that already appeared in Prop. 8 of the *DQA*, when it came to introducing the degenerate case into the ordinary one. But the strategy is completely different: there is no question of translating this kind of reasoning into a proof “in the manner of the Ancients”, but it is argued rather that its refusal would prevent some general equation from holding in ordinary algebra. The passage continues:

Thus we find in the calculations of ordinary algebra traces of the transcendent differential calculus and the same peculiarities about which some scholars have scruples. And even the algebraic calculus cannot do without them if it wants to conserve its advantages, one of the most considerable of which is the generality that is proper to it in order for it to comprehend all the cases, even the one where some given straight lines vanish. (GM IV 105)

Leibniz then mentions that the same argument, when used in Physics, is what he termed the “Law of Continuity” in the *Nouvelles de la République des Lettres* from 1687 (in the context of his polemics with Cartesians about the laws of motion). In this case, one relies on the introduction of the fiction of nascent and evanescent quantities, “taking equality for a particular case of inequality, and rest for a particular case of motion”. It is crucial to note, however, that Leibniz immediately recalls that one can easily translate this procedure into an “Archimedean” procedure by relying on the *PUD*, that is,

by supposing not that the difference between magnitudes that become equal is already nothing, but that it is in the act of vanishing, and the same with motion, which again is not absolutely nothing, but something that is on the point of being nothing. And if someone is not content with this, we can make him see, in the style of Archimedes, that the error is not at all assignable and cannot be given by any construction. (GM IV, 105).

The same kind of reasoning is repeated at the beginning of a text to which we shall return in greater detail later on, the *Defense du calcul*. In it, Leibniz first announces he will provide “a quite palpable means of justifying our ways of calculus by means of the ordinary calculus of Algebra”.[[78]](#footnote-78) He then takes up exactly the same example as in the *Justification*[[79]](#footnote-79) (changing only the lettering) and announces: “As regards the first point, I will show that without noticing it enough, in the ordinary Calculus of Algebra one has long been practising the method contested in relation to infinitesimals, when one applies a general calculus to some particular cases, where some magnitudes vanish”.

Following this thread, a new strategy for justifying the differential calculus against the attacks of algebraists such as Michel Rolle appears clearly: instead of trying to convince them that the apparent use of infinitesimals was harmless, one could posit the Law of Continuity as a postulate (common to ordinary algebra and Differential Calculus) and show that the rules of the calculus could be derived upon this supposition.[[80]](#footnote-80) This is exactly what the famous *Cum Prodiisset* proposed.[[81]](#footnote-81)

Some commentators have interpreted this new strategy as forcing a specific interpretation in terms of the existence of infinitesimals.[[82]](#footnote-82) This is a typical confusion between the issues of use and existence. Leibniz, for his part, was very clear about the neutrality of this new strategy as regards the question of existence, as was the case for his previous strategies also. We just saw that he mentions in the *Justification* that everything he said about vanishing quantities can be translated into a proof in the style of Archimedes by using the *PUD*. But it is especially in the *Defense* that he comments about the two possible interpretations of the “foundations” (“*fondement*”) he just presented:

the foundation of all of this can be explained by taking *z* and *x*, i.e. *ba*, *hA*, *in the very act of vanishing* and falling into *A*. It is like in a nascent motion, since an instant of motion is different from an instant of rest, and in this element of time, there is an element of nascent progress, which is more than nothing. But even if the instant of this act of vanishing or emerging (*naissance*) were in metaphysical rigor only a fiction (in order to escape the labyrinth *de compositione continui*), it suffices that no error could emerge from it, and that these fictions could always hold in place of truths in the calculus, in a like manner to imaginary roots. *Since rejecting all the infinitely smalls, and taking in their place only magnitudes as small as one wishes, one will always show that the error would be less than any given error. That is to say that there is none*. [Our emphasis]

Hence we cannot agree with the reading of this text as presenting a “semantic” strategy, which would amount to the acceptance of a model in which infinitesimals are introduced as genuine entities. Even if this interpretation of the strategy were possible, in both texts Leibniz emphasizes the fact that it is not necessary. By describing the strategy in terms of semantics, one is just forcing this necessity into a picture in which Leibniz thought and said it was *not* holding. If the duality of possible interpretations were not clear enough, one would just have to pursue the reading of the *Defense*: “This is what makes me speak on other occasions of incomparables, because what I say of them has its place whether one understands infinitely small magnitudes or one employs magnitudes of a smallness that cannot be considered and is sufficient to make the error less than that which is given.”

Proponents of the view that Leibniz held infinitesimals to be ideal elements in an extended continuum have drawn support from the fact that in describing his infinitesimals as incomparables, Leibniz makes reference to proposition 5.4 of Euclid’s *Elements* (5.5 in the editions of Euclid available to Leibniz). Thus in explaining his conception of incomparables to Nieuwentijt in his famous letter of 1695, Leibniz writes:

I believe those things are equal not only whose difference is absolutely zero, but also those whose difference is incomparably small; and even though this difference should not be said to be nothing at all, there is however no quantity comparable with them whose difference it is. … For only those homogeneous quantities are comparable, I hold with Euclid 5.5, one of which can be made greater than the other when multiplied by a finite number. (GM V 322)

Commenting on a similar passage in his letter to L’Hôpital,[[83]](#footnote-83) Katz and Sherry claim that in these passages “Leibniz describes such entities [infinitesimals] as ‘incomparable quantities’ and defines them in terms of the violation of what today is called the Archimedean property.” (Katz and Sherry 2012, 1555).[[84]](#footnote-84) But Leibniz has not thereby defined infinitesimals as non-Archimedean elements of an extended continuum. What he has defined is what it is for one quantity (say, *dx*) to be incomparable *in relation to* another (say, *x*). Formally, Leibniz’s definition gives:

For a given *x* and *dx*, *dx* INC *x* iff ¬(∃*n*)(*n dx* > *x*)

This figures, because if *x* and *dx* were comparable magnitudes, then there would be an assignable ratio between them in accordance with the Archimedean Axiom, so that (∃*n*)(*n dx* > *x*). But this definition precisely does not commit you to the *existence* of non-Archimedean infinitesimals *dx* in the continuum *x*, which would require:

For a given *x*, (∃*dx*)(∀*n*)(*dx* < *x/n*)

—or equivalently, for a given *x*, (∃*dx*) ¬(∃*n*)(*n dx* ≥ *x*).

But Leibniz makes no such assertion of the absolute existence of infinitesimals. Rather, as he explains to Luigi Grandi, infinitesimals are “relative nothings”,[[85]](#footnote-85) in that *dx* is nothing compared to *x*, although not compared to *dy* (if *y* is a function of *x*). And of course the same goes for *ddx* relative to *dx*, as opposed to relative to *ddy*, etc. And the justification for holding that *dx* is nothing relative to *x* is provided by the *Lemmata incomparabilium* and fully in accord with Archimedean principles, as the continuation of the above-quoted passage makes clear:

And those things that do not differ by such a quantity I hold to be equal, as Archimedes also assumed, as have all others after him. And this is the very thing which is said to be a difference smaller than any given. And, by a process that is indeed Archimedean, the matter can always be confirmed by an inference *ad absurdum.* But since the direct method is easier to understand and more useful for discovery, once this way of reducing is known it suffices that afterwards the method is applied in which incomparably small things are neglected, which certainly also brings with it its demonstration according to the lemmas communicated by me in February 1689. (GM V 322)

This is in complete accord with the use Leibniz makes of infinitesimals (and points at infinity) in his Law of Continuity.[[86]](#footnote-86) Here it does not have to be accepted that they are contradictory in order for them to be used successfully, since their use as “relative nothings” is justifiable by appeal to the Lemmas on Incomparables, and ultimately, the *PUD*. What this means is that nothing can be inferred about the existence of infinitesimals from their appearance in the surface language. As far as mathematics is concerned, they may be used (under the conditions stated) as if they exist. If one wants to infer existence, one cannot just rely on the nominal definition of “incomparables” (as not respecting the definition of Archimedean quantities), but one has to establish the possibility of such objects, i.e. that they are non-contradictory. But, as we have shown, Leibniz always claimed that infinite entities, be they infinitely large or infinitely small, could not be considered as genuine quantities without violating a constitutive property of quantities given by the part-whole axiom. Hence, they cannot be introduced into the system without contradiction. To show that one can build a consistent model including such infinite entities is not a way of “vindicating” Leibniz’s point of view. It is a way to show that, from a modern point of view, he was wrong in assuming that these infinite entities were contradictory.

A stronger argument arises when supporters of the non-standard view object that in fact all of this is pure wishful thinking on the part of Leibniz and that the kind of justification of the differential algorithm which is sketched in texts like the *Cum Prodiisset* under the postulation of the Law of Continuity, *cannot* be translated into a justification compatible with the syncategorematic view. In this sense, one “vindicates” Leibniz’ approach by showing that, even if he thought of infinitesimals as non-existing, he had to commit himself to an “extended continuum” to support the kind of justification he provided through the Law of continuity. This is the objection that we will address in the last part of our study.

So far, we have dealt mainly with the justifications of the *use* of infinitesimals and we have not yet tackled the crucial issue of the justification of the differential algorithm in and of itself (i.e. the justification that its rules are sound and trustworthy). As we have hopefully shown in the preceding developments, one benefit of proceeding in this way is precisely to disentangle issues that are too often confused (typically that of use and that of existence). More importantly, it clarifies what remains to be done. As we have indicated, Leibniz finally elaborated two strategies for the defence of the use of infinitesimals, which are presented in turn in the texts he sent in 1701-1702: the first one is a translation of the discourse using the fictions (of infinitely small or infinitely large quantities) into proofs in the manner of the Ancients (the prototype of this strategy being given by prop. 8 of the DQA);[[87]](#footnote-87) the second one relies on the Law of Continuity taken as a postulate, which should be accepted by both parties. In both cases, we have emphasized, Leibniz claims that the question of existence is not settled and that one can believe in real infinitesimals or not.

What remained to be done was a justification of the rules of the algorithm presented in 1684 following these two strategies.[[88]](#footnote-88) Interestingly enough, this is precisely what we find in the known texts from the same period. But there is more, as we will now demonstrate: these justifications were both compatible with the syncategorematic view (which, we insist, is a thesis about existence). The first strategy is typically the one expounded in the famous letter to Wallis from March 1699 and relies on the *Lemmata incomparabilium*. The second strategy is the one expounded in the *Cum Prodiisset* and relies on the *Lex continuitatis*. Let us take a detailed look at each of these texts in turn.

As we have seen, Leibniz’s justification strategy is consistent with the fact that “what is concluded by means of the infinite and infinitely small can be evinced by a *reductio* *ad absurdum* by my method of incomparables”.[[89]](#footnote-89) It is of first importance to notice that in all of these passages, Leibniz does not mention only *one* of these two foundations. The kind of justification presented in the DQA, as noted earlier, does not transfer immediately to the Differential Algorithm. On the other hand, as Leibniz regularly emphasizes, the *Lemmata incomparabilium* are neutral, in and of themselves, as regards the question of the complete eliminability of infinitesimals. It is therefore the connection of *both* principles which supports, according to Leibniz, a justification of the differential algorithm in which infinitesimals can be “evinced”. This strategy is clearly presented to Wallis in 1699. The whole exchange between the two authors, beginning in 1695, is very interesting.[[90]](#footnote-90) We cannot enter into all of its details here, but would like to emphasize that, once again, there are good reasons why Leibniz did not present the strategy in terms of “nascent” quantities to Wallis. Indeed, Wallis agreed about this model, but thought that one could infer from it that these nascent quantities were, at the end of the day, absolute nothings:

For my quantity *a* is the same as your *dx*, except that my *a* is nothing and your *dx* infinitely small. Then when those things are neglected which I hold should be neglected in order to abbreviate the calculation, that which remains is your minute triangle, which according to you is infinitely small, but according to me is nothing or evanescent. (Wallis to Leibniz, 30 July 1697; GM IV, 37, quoted and translated by Jesseph 1998)

The method Wallis is here alluding to is his own method of tangents, which is an improvement on Fermat’s method that he thinks to be fully equivalent to Leibniz’ s. As some readers of Fermat did at the time, he introduced the increments into the calculation and then, in the last step where the limit case was reached, discarded them as absolute zeros.

According to Wallis, what the “characteristic triangle” stood for was a triangle whose shape was preserved in the process, but not the magnitude (since it became a triangle of magnitude “zero”).[[91]](#footnote-91) In the limit case, what remained henceforth was a shape without magnitude. Leibniz noticed that this idea was obscure. More, profoundly, he made Wallis notice that taking the *dx* to stand for an absolute nothing would prevent the possibility of taking its difference and the difference of its difference, etc.

Of course, the form of the characteristic triangle can be rightly explained by the degree of declination, but for the calculus it is useful to imagine [*fingere*] quantities infinitely small, or as Nicholas Mercator called them, infinitesimal: and such things cannot be taken for nothing when the assignable ratio among them is sought. On the other hand they are rejected whenever they are added [*adjiciuntur*] to quantities incomparably greater, according to the lemmas on incomparable quantities I once proposed in the *Acta Eruditorum* of Leipzig, which foundation the Marquis de L'Hôpital also uses. (Leibniz to Wallis, 30 March 1699; GM IV, 63)

Quoting this passage, D. Jesseph concludes that Leibniz is here lining up on the side of the Marquis de l’Hôpital and hence as a defender of the reality of the infinitesimals.[[92]](#footnote-92) But, strangely enough, he fails to quote the rest of the passage in which Leibniz departs from l’Hôpital by explaining how the infinitesimals could be evinced by a combination of the *Lemmata* and of a proof “in the style of Archimedes”:

…there remains *xdy + ydx + dxdy*. But this *dxdy* should be rejected, as it is incomparably smaller than *xdy + ydx*, and this becomes *d(xy) = xdy + ydx*, inasmuch as, if someone wished to translate the calculation into the style of Archimedes, it is evident that, when the thing is done using assignable qualities, the error that could accrue from this would always be smaller that any given.

Now, in order to make the whole procedure fully clear, let us write out this tedious Archimedean proof.

What the *Lemmata incomparabilium* tell us is where to look for the “difference” which has to be made “smaller than any given error”. It is a crucial issue since the whole algorithm is dealing with “differences”, and a main reason why the transfer of reasoning in the style of the DQA is not immediate is that we need to identify which of these “differences” will be considered as the “error” to render equal to zero.[[93]](#footnote-93) In the case under study, considering that we are dealing only with assignable quantities, the *Lemmata* indicate to us that the “error” will be located in the *dxdy*, which is incomparably smaller than *dx* and *dy*. We then just have to suppose that the equality does not hold and that *d*(*xy*) differs form *xdy* + *ydx* by a given quantity Z. Now, since we can take *dx* and *dy* as small as we wish, let’s take them smaller than 1/4Z. We hence have *d*(*xy*) = *xdy + ydx + dxdy* < *xdy + ydx* + 1/16 Z. And since *d*(*xy*) was supposed to be equal to *xdy + ydx* + Z, we obtain that Z < 1/16 Z. *Quod est absurdum*. The same reasoning applies, evidently, to all the rules of the algorithm and is just a matter of boring routine.

Let us now turn to the second strategy, that which relies on the Law of Continuity and is presented in the *Cum Prodiisset*. Here what Leibniz has to do is simply to generalize the kind of example that he expounded in the *Justification* (and which may have been chosen for that precise reason). The core of the strategy is henceforth to rely on proportionalities, typically the one given by the ratio between the “characteristic triangle” of sides *dx*, *dy* and the triangle made by the ordinate and the subtangent.[[94]](#footnote-94) This calls for a different approach than in the letter to Wallis since we then need to reach an equation in which the proportion *dy/dx* is made apparent. This is particularly salient in the first example which Leibniz proposes to tackle, the finding of the tangent to the parabola of equation *xx = ay* (or y = *xx:a*). By introducing the increments *dx* and *dy*, one obtains: *y + dy = xx + 2xdx + dxdx,:a*.[[95]](#footnote-95) But Leibniz does not remain content with subtracting *y* and *xx:a* from both sides. He also divides both sides by *dx* to obtain:

*dy*:*dx* = *2x + dx*,:*a*

which is the general rule expressing the ratio of the difference in ordinates to the difference in abscissas. That is, if the chord *1Y2Y* is produced until it meets the axis in *T*, then the ratio of the ordinate *1X1Y* to *T1X*, the intercepted part of the axis between the point of intersection and the ordinate, will be as *2x + dx* to *a*.

Up to this point, we have been dealing only with finite quantities and this is where the Law of Continuity will now enter the scene to conclude the reasoning:

Now, since by our postulate it is permissible to include under one reasoning also the case where the ordinate *2X­­2Y­­*, having been moved closer and closer to the fixed ordinate *1X­­1Y­­* until it finally coincides with it, it is clear that in this case *dx* will be equal to zero ôr should be omitted, and so it is clear that, since in this case *T1Y* is the tangent, *1X­­1Y­­* to *T1X* is as 2x to *a*.

This first basic example was not mentioned in Henk Bos’s seminal study and this had the unfortunate consequence of hiding the global structure of the reasoning. Indeed, after taking another example (the curve of equation *x3 = aay*) and commenting on the fact that the differential calculus, *like ordinary calculus*, “applies equally to the case where the difference is something and to where it is zero”, Leibniz continues:

But if we want to retain *dx* and *dy* in the calculation *in such a way that they denote non-vanishing quantities even in the ultimate case*, let *(d)x* be assumed to be any assignable straight line whatever; and let the straight line which is to *(d)x* as *y* ôr *1X­­1Y­­* is to *1X T* be called *(d)y*, so *dy* and *dx* will always be assignable to one another in the ratio *D2Y* to *D1Y*, which latter vanish in the ultimate case. [our emphasis]

This second method differs from the first one in the sense that it *does not rely on vanishing quantities*. Moreover, it has the particularity of taking *(d)x* as constant.[[96]](#footnote-96)

Here again, the presentation by Bos, as thorough and interesting as it is, had the unfortunate consequence of hiding some particularities of the reasoning by neglecting some developments that were apparently of no interest. Typically, he did not detail the boring justification of the rule for subtraction given by Leibniz. But this justification, as obvious as it may be, is nonetheless telling, since it arrived at the following equation:

*(d)y –(d)z,:(d)x = (d)v:(d)x*. And so *(d)y* – *(d)z* = *(d)v*, as was proposed

In this equation, there are only assignable quantities, a situation which should be intriguing for those who claim that what derives from the law of continuity *commits us* to the existence of real unassignable quantities. Moreover, Leibniz then comments:

Although we may be content with the assignable quantities *(d)y, (d)v, (d)z,* and *(d)x*, since in this way we can perceive the whole fruit of our calculus, namely a construction using assignable quantities, still it is clear from this that we may, at least by feigning, substitute for them the unassignables *dx, dy* by way of fiction even in the case where they vanish, since *dy:dx* can always be reduced to *(d)y:(d)x*, a ratio between assignable or undoubtedly real quantities.

It is pretty clear here that Leibniz contrasts the proof relying uniquely on assignable quantities, with the one in which one can introduce the fiction of vanishing quantities. The justification of this second discourse is that all the vanishing quantities stand for “ratios between assignable or undoubtedly real quantities”.

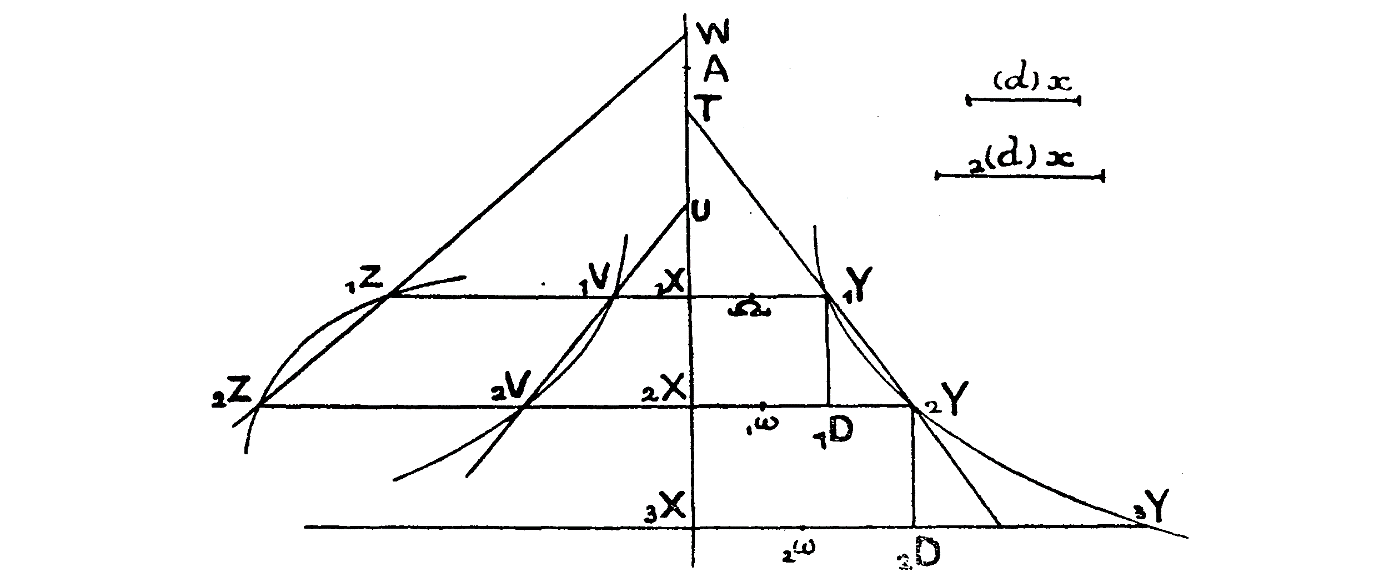
The situation is a little more intricate with the example of the derivation of the rule for the product *d(xv)*, the example chosen by Bos. Here Leibniz reached the following situation:

*a(d)y/(d)x = x(d)v/(d)x + v + dv*

so that the only remaining term which can vanish is *dv*, and since in the case of vanishing differences *dv* = 0.

It may seem that in this case the last equation involves irreducibly the existence of an unassignable quantity. But this is not the case. As in the strategy developed with Wallis, the whole interest of this procedure is to isolate the part of the equation which will constitute the difference to be rendered as small as one wishes. Moreover, the drawing (see fig. below) makes it clear to what it corresponds (the quantities other than *dv*, taken into quotients, correspond to finite ratios between ordinates and subtangents to the given curves). We henceforth are facing a typical “syncategorematic” interpretation in which all the “nascent” and “evanescent” quantities have been replaced by finite quantities which can be taken as small as one wishes.

Leibniz stops here, but he may have easily continued by providing another boring proof “in the style of Archimedes”: one just has to suppose that *(d)y/(d)x* differs from *x(d)v/(d)x + v* by a given error Z. Then by taking *d(x), d(y), d(v)* as small as one wishes, for example smaller than 1/4Z, and since all the other terms are finite quantities whatever the size of *(d)x*, *d(y)* may be, we easily reach the contradiction of having Z smaller than itself.



A last comment: as can be seen in both examples (Wallis and *Cum Prodiisset*), the rendering of the proof of the procedure into an “Archimedean” argument is completely trivial and this may explain why Leibniz never felt compelled to give it in detail. We have made it fully explicit only in order to counter the claim according to which it was only wishful thinking on Leibniz’s part, and that he would have been completely unable to provide such a proof.

6. Conclusion

To finish, let us go back to the *Defense du Calcul*. In the margin of the *Cum prodiisset*, Leibniz has written the following commentary (not translated by Child):

All this must be edited very carefully so it could be published, omitting what is harsher in contradicting others. It is to be joined by my method for the law of continuity shown by the tracing of lines, and also by the tract [*schediasma*] I had sent the Parisians in order to show that in the common example the ratio between nothings is feigned to be something.

It is highly plausible that the tract sent to the Parisians mentioned here is the *Justification*—not only because the descriptions of the contents match, but also because this is indeed the only *schediasma* which was sent by Leibniz to the Parisians (the other texts being in fact letters of which extracts were made in order to be published). Now the *Defense* is precisely a rewriting of the *schediasma/Justification* after an introduction which announces very nicely the content of the *Cum Prodiisset*. We tend to think that it was conceived as the planned edition of the *Cum Prodiisset* with the addition of the *Justification*. Be it as it may, the most interesting aspect of the *Defense* is the way in which it spells out the various issues at stake. For convenience, we will indicate these various stakes and Leibniz’ strategy for responding to them with numbers in the text:

I learn that talented people are opposed to the Calculus of Differences, (1) because it seems that in it one necessarily proceeds by infinitesimals, or by infinitely small magnitudes, and (2) because they believe that in it one makes elisions at pleasure. One can always show to them (1a) that everything that is concluded by this calculus can be proved by a reduction ad absurdum in the style of Archimedes, and by using the Lemmas of incomparables proposed in the Leipzig Acta; and (2a) it is always easy to recognize what one can neglect with impunity without any error arising from it, so that the elisions are made according to certain rules, and not as we see fit, except in that it is permitted to give to the continuously variable magnitudes differences as one wishes by choosing the progression one finds appropriate; this means that one can choose a series whose differences are constant, and whose second and other differences vanish. But without here making use of these Archimedean-style demonstrations, which are extremely long and not sufficiently appropriate to enlighten the mind, I want to propose here two things: (3a) a very palpable way of justifying our method of calculation by means of ordinary algebraic calculus; and then (4a) an interpretation of the Calculus of Differences where instead of infinitesimals one understands only very small magnitudes, and this does not stop one from reaching a conclusion (LH 35 VI 22 Bl. 1–2. 1. Pasini, 1988, p. 705).

This constitutes a beautiful summary of what has been shown in the previous development: Leibniz was consistent about the fact that one could *always* eliminate the reference to infinitesimals that his adversaries claimed to be necessary to his procedures. Moreover, he showed that the rules of his algorithm could be justified on that very basis. The recourse to the Archimedean procedure (plus the *Lemmata*) was the standard procedure to provide such a justification. But Leibniz also developed an alternative strategy in which the foundations of the calculus were justified by the fact that “ordinary algebra” proceeded on the very same foundations. Now, one could doubt if this second strategy was consistent with the first one in the sense that it was expressible without recourse to infinitesimals and this is what the last sentence expresses very clearly: one can produce an interpretation of the second strategy where “*in place of infinitesimals*, one understands only very small magnitudes and one no less reaches a conclusion” (our emphasis). This should be sufficient to achieve our demonstration: not only did Leibniz claim that one could justify the calculus without recourse to infinite quantities, but he provided precisely such a justification even in the case where the basic postulate was the *Lex Continuatis*.

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1. The question of the “existence” of mathematical entities is a tricky one since it can relate to various questions, including ontological problems concerning their existence “in nature” or the like. In this paper, we take mathematical existence in its strictest sense of non-contradiction and we say that an entity does not exist when its positing in the theory involves introducing a contradiction. [↑](#footnote-ref-1)
2. That this is not accurate can already be seen in the case of Abraham Robinson, who gives a justification of infinitesimals using finitist means (using his compactness theorem), without it following that they are therefore replaceable by finite terms. [↑](#footnote-ref-2)
3. For a clear and comprehensive explanation of the development of pre-Robinsonian non-Archimedean theories, see Ehrlich (2006), who gives the lie to Russell’s claim that infinitesimals had been banished from mathematics. [↑](#footnote-ref-3)
4. A particularly clear example of this tendency to dichotomize is provided by Mikhail Katz, David Sherry and various co-workers in a spate of recent articles. According to their analysis, there are two rival approaches in the history of infinitesimal mathematics, which they call the “A-track” and the “B-track”. The first is the Weierstrassian approach, with ‘A’ standing for the fact that the elements of the continuum obey the Archimedean axiom, whereas ‘B’ stands for the “Bernoullian continuum”, in which there are elements violating that axiom (non-Archimedean infinitesimals). [↑](#footnote-ref-4)
5. This must not be confused with the question of whether infinitely small things exist in nature; here also Leibniz denies that they do, holding that there are just arbitrarily small finite things. But this latter fact, Leibniz believes, is what allows the applicability of the ideal notions of mathematics to nature. More on this below. [↑](#footnote-ref-5)
6. As we will detail below, these criticisms are made in (Blåsjö 2017) and (Jesseph 2015) [↑](#footnote-ref-6)
7. “There are actually parts in the continuum … and these are actually infinite (*Sunt actu partes in continuo …et sunt actu infinita*)” (A VI 2, 264/LLC 339). Similarly in his *De minimo et maximo* (Nov 1672-Jan 1673) Leibniz asserted that “There are in the continuum infinitely small things (*Sunt aliqua in continuo infinite parva*)” (A VI 3, 98/LLC 12). [↑](#footnote-ref-7)
8. For example, Douglas Jesseph writes: “[Leibniz’s] view of infinitesimals as ‘useful fictions’ seems to have taken shape in the mid 1690s, although there are certainly traces of it as early as the 1670s, and a forthright statement of the fictionalist position seems to have come from Leibniz’s pen only in the aftermath of the dispute in the Académie des Sciences over the foundations of the calculus.” (Jesseph 2015, 198). On the contrary, a forthright statement of the fictionalist position from April 1676 is given at the head of this article; and the position is defended in the *De Quadratura* that Leibniz finished the same year. [↑](#footnote-ref-8)
9. See also Leibniz’s remark to Varignon, still in 1713, concerning Grandi’s series: “But one must not rely on reasonings about infinite series, unless one can demonstrate their truth with finite quantities by the methods of Archimedes. [*Mais il ne faut se fier aux raisonnemens sur les series infinies, que lorsqu'on en peut demontrer la vérité par les finis à la façons d’Archimède*].” (GM IV 191). [↑](#footnote-ref-9)
10. See, for example, *Historia et Origo Calculi Differentialis* (circa 1713), transl. (Child 1920, 29-30). [↑](#footnote-ref-10)
11. *Observationes quod rationes seu proportiones, non habeamt locum circa quantitates nihilo minores et de vero sensu methodi infinitesimalis* (GM V 389). This paper, which is presented as providing the “true meaning of infinitesimal methods” was published in the *Acta Eruditorum* in 1712. [↑](#footnote-ref-11)
12. “Our thinking about composites is for the most part only symbolic” (*Meditations on Knowledge, Truth and Ideas*, A VI 4, 588). [↑](#footnote-ref-12)
13. “For we often understand the individual words in one way or another, or remember having understood them before, but since we are content with this blind thought and do not pursue the the resolution of notions far enough, it happens that a contradiction involved in a very complex notion is concealed from us” (A VI 4, 588). This is why nominal definition is not enough and should be accompanied by a proof of possibility (we’ll encounter this question with the definition of “incomparable” quantities defined as violating Archimedes’ axiom). Cf. Arthur’s discussion of real versus nominal definitions in the context of how Leibniz might have responded to Cantor’s definition of the transfinite (Arthur 2019). [↑](#footnote-ref-13)
14. This is also what happens in *reductio* argument, except that in this case we focus on the part containing the contradiction to exhibit the impossibility involved in the fiction (the connexion with “impossible fictions” and *reductio* is made explicitly in the Correspondance with Clarke, see *Correspondance Leibniz-Clarke* IV, §16-17, 374). The parallel with a point at infinity may be recalled here since this is a notion which produces a contradiction when inserted in some proofs of Euclid’s *Elements* (such as I, 27, where we assume that parallel lines meet), but which is also useful (when accompanied with suitable demonstrations) in order to produce general geometrical truths, such as the ones promoted by Desargues and Pascal. [↑](#footnote-ref-14)
15. Cf. Leibniz’s remark in a letter to Foucher: “It is true that from truths one only infers truths; but there are certain falsities useful for finding the truth [Il est vray que des verités on ne conclut que des verités; mais il y a certaines faussetés utiles pour trouver la verité].” (GP I 406) [↑](#footnote-ref-15)
16. See, for example, the *De Conditionibus* (1665), § 164/307 (A VI, 1, 143). [↑](#footnote-ref-16)
17. See, for example, the *Specimen inventorum de admirandis naturae Generalis arcanis* (1688 ; A VI 4, 1628/ LLC 327-9), or the *De abstracto et concreto* (1688 ; A VI 4, 991). [↑](#footnote-ref-17)
18. See the passage from *Observationes quod rationes* quoted above, or the Letter to Wolff : « Dès ce moment et de manière paradoxale et pour ainsi dire par une *Figure Philosophico-rhétorique* nous pouvons considérer le point par rapport à la ligne, le repos par rapport au mouvement, comme des cas particuliers compris dans le cas général inverse, le point apparaissant comme une ligne infiniment petite, évanescente, ou le repos comme un mouvement évanescent. De même pour d’autres formules du même genre, que l’homme très profond qu’était Joachim Jung aurait nommées vraies par tolérance et qui sont des plus utiles pour l’art d’inventer, même si à mon avis elles enveloppent quelque chose de fictif et d’imaginaire » (Lettre à Wolff, 1713, GM V, 382-387) [↑](#footnote-ref-18)
19. Another important aspect of this parallel is that there exist non-standard Set theories, provably equiconsistent with the standard one, in which one introduces a two-tiered ontology with classes as genuine entities. The standard set theorist can henceforth claim at the same time that proper class do not exist (in her axiomatic system) and that her surface language remains neutral as regard this question of existence since there are other ways of interpreting it (as recalled by Kunen on that very same page). [↑](#footnote-ref-19)
20. It is worth emphasizing that the premise of the whole discussion of this important paper, and of the immediately preceding “On Motion and Matter”, is Leibniz’s “recent discovery” that entities he had previously taken as actually infinitely small, such as horn angles and endeavours, are not so after all, and must be classified as fictions. See (A VI 3, 492; LLC 75, 394, 396). [↑](#footnote-ref-20)
21. Cf. Leibniz’s comment in his 1675-6 treatise *De Quadratura* that he prefers a justification “which simply shows that the difference between two quantities is nothing, so that they are then equal (whereas it is otherwise usually proved by a double reductio that one is neither greater nor smaller than the other)” (Leibniz 1993, 35). More on this below. [↑](#footnote-ref-21)
22. For an introduction of this principle and the comparison with Newton see (Arthur 2008). See also Levey’s paper in the same volume: he calls a similar principle Leibniz’s “Principle of Equality” (Levey 2008, 113). [↑](#footnote-ref-22)
23. Interestingly enough, in the exchange with Bernoulli, Leibniz refers to the fact that he proved, long ago, that an infinite number is a contradictory notion: “*Sane ante multos annos demonstravi numerum seu multitudinem omnium numerorum contradictionem implicare…*[Of course, I demonstrated many years ago that the number or multiplicity of all numbers implies a contradiction]” (A III 7, 884) This letter also includes, as we shall see, an important reference to the *DQA* of which Leibniz copies the argument on the area under the hyperbola and concludes by the fact that one could use infinitely small quantities as fictions as long as one can demonstrate them by using *reductio ad absurdum* and his *Lemmata incomparabilium* (A III 7, 857 and below section 4). [↑](#footnote-ref-23)
24. In the seventeenth century it was customary to refer to “the syncategorematic infinite” as an abbreviation for “the infinite understood syncategorematically. For a thorough treatment of the Scholastics’ discussions of the syncategorematic, see Sara L. Uckelman’s (2015). Leibniz also recognized a third species of the infinite, namely the hypercategorematic infinite. See (Antognazza 2015) for discussion. [↑](#footnote-ref-24)
25. Compare with what Leibniz wrote to Des Bosses in 1706: “I hold that matter is actually fragmented into parts smaller than any given, that is to say, that there is no part that is not actually subdivided into other parts undergoing different motions. This is demanded by the nature of matter and motion, and by the whole frame of the universe, for physical, mathematical and metaphysical reasons” (2 March 1706; LDB 32-35) [↑](#footnote-ref-25)
26. Cf. what Leibniz wrote to Samuel Masson in the last year of his life: “Notwithstanding my Infinitesimal Calculus, I do not at all admit a genuine infinite number, although I confess that the multiplicity of things surpasses every finite number, or rather, every number.” (GP VI 629) [↑](#footnote-ref-26)
27. The best that could be done, and was done by Pascal and Torricelli, is to show on examples that the same results were obtained either by using indivisibles or by relying on the “method of the Ancients”. [↑](#footnote-ref-27)
28. This is typically the case in Pascal and Wallis, who both believed “indivisibles” to stand for infinitely small homogeneous quantities (Pascal is explicit about the fact that one takes “des petites portions égales”). [↑](#footnote-ref-28)
29. As we will see below, this qualification was explicit when Leibniz presented a variant of the proof to Bodenhausen in 1690. [↑](#footnote-ref-29)
30. One just has to fix a point as origin and take triangles under the first curve from this point, each triangle being transformed by the general construction into a rectangle of which one can estimate the difference with the second curve. This is crucial in the demonstration for two reasons: first, it enables one to give a proof which works in general (there are, of course, some conditions on the curve, the first one being that it has a tangent at every point, but Leibniz makes these constraints explicit, see Rabouin 2015) ; second, it allows one to transform the “difference” between the triangulation and the curve comparable with a well known figure, of which one can estimate the magnitude (in this case a rectangle, one brilliant idea of Leibniz being the use of an upper bound for the width of this rectangle). [↑](#footnote-ref-30)
31. See (Knobloch 2002, 65) and (Rabouin 2015, 356) for details. [↑](#footnote-ref-31)
32. Leibniz mentions the need for “rigour” in several places in the *DQA*. Thus at the beginning of Prop. 6 he writes “The reading of this proposition can be omitted if in demonstrating Prop. 7 one does not desire the utmost rigour [Hujus propositionis lectio omitti potest, si quis in demonstranda prop. 7. summum rigorem non desideret].” Elsewhere he writes of *severas demonstrationes*, and *severe demonstrar*e. [↑](#footnote-ref-32)
33. This idea already appears – for the first time, it seems – in Pascal, see (Whiteside 1968, 341) and (Cortese & Rabouin 2019). [↑](#footnote-ref-33)
34. The validity of this general result on sum of “differences” has been demonstrated by Leibniz in Prop. 5. [↑](#footnote-ref-34)
35. As we will see later, Leibniz still insists on this in the *Compendium* of 1690, when commenting on the *DQA* and warning about the danger of reasoning with the infinite: “*In Hyperbola Conica, quia zona aequalis zonae conjugatae, fiet totum aequali partem. Unde patet rem reducendam ad demonstrationes apagogicas* [In the hyperbolic conic, since a zone is equal to a conjugate zone, this makes the whole equal to a part. Whence it is clear that the matter should be reduced to apagogical demonstrations.]” (GM V, 106). [↑](#footnote-ref-35)
36. Interestingly enough, when Leibniz rewrites the proof of Prop. 7 for the *Compendium* (to which we shall return in section 4), he formulates it in a mixed way: starting as in a *reductio* by supposing that the intended result does not hold (i. e. that the *Quadrilineum* is not twice the *Trilineum*), and then by supposing that there is a difference Z between the two quantities, he demonstrates that the difference can be made smaller than Z But instead of directly concluding *quod est absurdum*, he continues: “*erit per prop. 5 diff. inter Q et T minor quam 3/4Z, adeoque minor quam Z, adeoque minor data quacunque quantitate, adeoque nulla est haec differentia* [By prop. 5 the difference between *Q* and *T* will be smaller than 3/4*Z* and so smaller than *Z*, and so smaller than any given quantity whatever, and so this difference will be zero].” (GM V, 101). We see here very clearly the way in which any *reductio* of this type can be transformed into a direct argument using PUD and vice versa. [↑](#footnote-ref-36)
37. For some hypotheses in this direction see (Rabouin 2015). [↑](#footnote-ref-37)
38. According to Whiteside (see note 27 above), Pascal is the first author to have the idea of reasoning directly on the difference (which can be made smaller than any given quantity). In the *Lettres d’A. Dettonville* he applies this reasoning to the case of the division of the basis into a “sum of lines” (i.e. the basic techniques of the “method of indivisibles”), (Pascal 1659, 10-11). [↑](#footnote-ref-38)
39. This aspect should not be underestimated. According to Whiteside, it was one of the major obstacles which prevented early modern authors from correctly assessing the power of exhaustion methods, when taken in their logical form (Whiteside 1968, 331). [↑](#footnote-ref-39)
40. “*As regards this proposition itself*, I hold that it is one of the most general and useful in geometry, inasmuch as it is so general that it applies to all curves, even in the case where they are drawn at will without a specific law; and given any figure it exhibits infinitely many others, the dimension of each of which depends on the former and vice versa. But it can also be reckoned among the most fruitful theorems in geometry; for from it are demonstrated the quadratures of all parabolas and hyperbolas to infinity (…) and not to mention that we have transformed infinitely many other absolute or hypothetical quadratures—certainly the circle and by its aid any conic having a centre—into a rational figure, and from this we derive a rational quadrature of the whole circle and any portion of it, and a true and perfect analytic expression for the arc from a given tangent, all of which it is the business of this treatise to demonstrate” (Scholium to Prop. 7, Leibniz 1993, 35-46, Leibniz 2016, 38). In the following paragraph, Leibniz insists on the fact that he can use triangles or rectangles to perform quadratures). [↑](#footnote-ref-40)
41. The idea of fiction is mentioned a first time for designating the “point at infinity” introduced by geometers developing projective considerations, such as Desargues and Pascal (schol. VII). We shall return to that example later. [↑](#footnote-ref-41)
42. In the *Compendium Quadraturae Arithmeticae* Leibniz defines a *segment* as “the space comprised between two lines, a curve, and another straight line”, and a *sector* as “a *trilineum* comprised between two straight lines and a curve” (GM V 101). [↑](#footnote-ref-42)
43. On this geneaology, see (Grosholz 2007, chap. 8.2). [↑](#footnote-ref-43)
44. « Pour y comprendre la Parabole et la ligne droite il faut se servir des lignes infinies et infiniment petites. Or posons que la ligne, q, ou le *latus transversum* de la Parabole soit d’une longueur infinie, il est manifeste, que l’Equation 2axq ± ax2 = qy2 , sera equivalente à celle cy : 2axq = qy2, ou 2ax = y2 (qui est celle de la Parabole) parce que le terme de l’Equation ax2, est infiniment petit, à l’égard des autres 2axq, et qy2, car puisqu’il y a autant de lettres ou dimensions d’un terme, que de l’autre, ceux dont une lettre est infinie, seront infiniment plus grands, que celuy dont les lettres ne sont qu’ordinaires : qui par consequent pourra estre negligé, puisque l’erreur qui en proviendra ne sera qu’infiniment petite, ou moindre qu’aucune erreur donnée, c’est à dire nulle. » (chap. XLIV. Notations modernized. A VII, 7, 103-104). [↑](#footnote-ref-44)
45. « *il faut en cas de la coincidence des points B et C concevoir la ligne BC infiniment petite, àfin que l’equation ne contredise pas l’egalité entre AC et AB*. » (A VII 7, preprint p. 60). The example of the coincidence taken as infinitely small distance is mentioned in the letter to Varignon from Feb. 2 1702 as a typical case of application of the Law of continuity (A III 9, 14). [↑](#footnote-ref-45)
46. We’ll see that this point was at the core of the discussion with Wallis in 1699. [↑](#footnote-ref-46)
47. This proposition is crucial for dealing with the quadrature of the simple hyperbola, a case which Leibniz deals with in prop. 12. [↑](#footnote-ref-47)
48. This presupposes that the curve is, a least piece wise, convex—a condition which stated by Leibniz when describing the curve in Prop. 6. [↑](#footnote-ref-48)
49. (Knobloch 2002) has given other examples, which can be generalized from prop. 20. [↑](#footnote-ref-49)
50. See (Mancosu & Vailati 1991). Leibniz also mentions other examples of the same kind in Gregory of Saint Vincent and Huygens. [↑](#footnote-ref-50)
51. See the beginning of the *Cum Prodiisset*, (Leibniz 1846, 41). The parallel between the introduction of point at infinite distance and infinitesimals, which appears in the *DQA* and was already present in Desargues, plays also a prominent role in the *Elementa nova matheseos universalis* (circa 1683; A VI 4, 521) and the *Matheseos Universalis pars prior* (1699; GM VII, 75-76). [↑](#footnote-ref-51)
52. This consideration will become even more significant when Leibniz realizes that the main supporters of the Differential Calculus in the Académie Royale (the Bernoullis, L’Hôpital and Fontenelle) all believe in the existence of infinitesimal quantities. [↑](#footnote-ref-52)
53. Especially when it comes to interpreting the “incomparables”, since what Leibniz expressly says is the following: “C’est ce qui m’a fait parler autres fois des incomparables, par ce que ce que j’en dis a lieu *soit qu’on entende des grandeurs infiniment petites ou qu’on employe des grandeurs d’une petitesse inconsiderable et suffisante pour faire l’erreur moindre que celle qui est donnée*” (*Defense du calcul* in (Pasini, 1988, 708; our emphasis). [↑](#footnote-ref-53)
54. A similar point was made in by Tzuchien Tho in his (2012), where he distinguished the question of the elimination of reference to infinitary terms by syncategorematic paraphrase from that of how Leibniz conducted his proofs in the DQA.. [↑](#footnote-ref-54)
55. Although Leibniz did not use this language in props. 6 and 7, this is what he alludes to when introducing the fiction of “infinitely small” at the beginning of Prop. 8: *Hoc uno verbo confici potest, ex eo quod quae p r o p o s i t i o n e 7. Demonstravimus generalia sunt, et locum habent, utcunque parvae sint rectae, ac proinde etsi sint infinite parvae*. [↑](#footnote-ref-55)
56. Amongst other problems, one difficulty is that it is not easy to find an equivalent of a generic case. [↑](#footnote-ref-56)
57. Victor Blåsjö writes: “it is well known that Leibniz was desperate to fashion a career for himself in intellectual circles at this time. The fact that he wanted to submit his work to the French Academy could very well be a reflection of this desire more than an assessment of the quality of the work, so this in itself proves nothing.” (2017, 136). Doug Jesseph “suspect[s] that he set aside the *Arithmetical Quadrature* without publishing it because he had turned his attention to more powerful methods that he would introduce in the 1680s…” (Jesseph 2015, 200). [↑](#footnote-ref-57)
58. Thus Jesseph writes that Leibniz’s procedure in the DQA, in its dependence on the construction of auxiliary curves, “requires that we have a tangent construction that will apply to the original curve”, and that although “this is readily available in the case of the circle, and tangents to conic sections and other well-behaved curves are also constructible with classical methods” (Jesseph 2016, 199), this would not extend to more general curves for which no geometrical tangent construction was available. [↑](#footnote-ref-58)
59. (Blåsjö 2017, 136). We will return to this below. [↑](#footnote-ref-59)
60. Many of these remarks denigrating the treatise depend on the bias we noted above concerning the concentration on Proposition 6 to the exclusion of what else is contained in it. When considered in its entirety, the DQA contains many beautiful results which Leibniz had not published at the time and he will still praised in later period: an easy way to find the quadrature of the cycloid, his famous series for the quadrature of the circle, a unified and analytic treatment for trigonometric functions, a study of the logarithmic curve, the presentation of his “harmonic triangle”, a proof of the impossibility of an algebraic quadrature of the circle. By the time he wrote to Bernoulli, it is true, these would not have carried the same weight, since all of this was then published in other places. But as we shall see, this did not derogate from the importance of Proposition 6 for Leibniz. [↑](#footnote-ref-60)
61. See in particular A III 4, 520 which Leibniz entitles: *Responsio mea ad dubium hujus Epistolae*. Among the difficulties Bodenhausen raises is a case where Leibniz concludes from the equation *eg = m* (where *e* is extension and *m* is mass) that *d(e)g=d(m)*, even though *g* is not a constant. In a lengthy and intriguing reply, Leibniz writes that while *e* and *m* are sometimes *assignabiles* and sometimes *inassignabiles*, *g* is introduced by “a kind of fiction” (A III 4, 523). [↑](#footnote-ref-61)
62. The passage begins with the following sentence: *Ad proferendum aliquod plausibile specimen nostrorum inventorum Geometricorum, quod ad captum sit eorum, qui veterum Methodis unice assueti sunt, non incommodum erit Theorema generale*…(A III, 4, 624). [↑](#footnote-ref-62)
63. The proof of Prop. 6 introduces an interesting variant by considering not the chords to the curve, but the triangles. This allows Leibniz to recover a technique by inscribed and circumscribed figures, which he presents as “Archimedean” (A III, 4, 635) [↑](#footnote-ref-63)
64. *Fateor autem me Theorematis hujusmodi opus non habere, nam quicquid ex illo duci potest, jam in calculo meo comprehenditur* (and this is what he will intend to show by rephrasing prop. 8 in the Differential calculus). Notice however that Leibniz adds that he still enjoys this method for it gives some representation in imagination corresponding to the operations of the Calculus (*libenter tamen iis utor. quia calculum imaginationi quodammodo conciliant*). [↑](#footnote-ref-64)
65. « *De Tetragonismo meo placet difficultatem quae viro eximio occurrerat, sublatam esse, sed quia demonstrationem desiderare videtur, fundamenta hic apponam, ex quibus eam ipse facile absolvet, nam cuncta nunc prolixe explicare non vacat. Habeo quidem plures demonstrandi idem vias, sed haec maxime elegans visa est*. » This passage begins the second half of the enclosure Leibniz sent to Otto Mencke for Sturm in November-December 1695, omitted from previous editions that had included the first half, such as Dutens (1768) and Erdmann (1840). [↑](#footnote-ref-65)
66. *Elemens des mathematiques ou principes generaux de toutes les sciences qui ont les grandeurs pour objet,* Paris 1675. [↑](#footnote-ref-66)
67. *Quadratrix autem si esset parabola, daretur sectio anguli in data ratione, per certi gradus aequationem, quod est impossibile, cum altior sit aequatio prout arcus vel angulus in plures partes secari debet* [“But the quadratrix, if it were a parabola, a section of the angle in the given ratio would be given by an equation of a certain degree, which is impossible, since the equation would be higher [in degree] in proportion as the arc or angle would have to be cut into several parts.]”] (A II 3, 104). [↑](#footnote-ref-67)
68. Recall that this is also the context in which he mentions his syncategorematic view on the infinite as grounding his conception of “fictions”. [↑](#footnote-ref-68)
69. See above, p. ???. [↑](#footnote-ref-69)
70. The expression is used by Varignon in a letter to Bernoulli, giving us a *terminus a quo* for these debates: (A Jean Bernoulli du 6 août 1697, *Der Briefwechsel von Johann Bernoulli*, Band 2, op. cit., p. 124). [↑](#footnote-ref-70)
71. Clüver was in fact criticizing Archimedes on a par with the new calculus, see (Mancosu 1996, 157). [↑](#footnote-ref-71)
72. To Mersenne, 9 January 1639 (AT II, 490). Leibniz is familiar with this letter and recalls it strategically at the beginning of the *Cum Prodiisset*, adding that Descartes did indeed rely on an Archimedean argument in his Metaphysics (Leibniz 1846, 42). [↑](#footnote-ref-72)
73. Letter to Hardy (AT I, 490). [↑](#footnote-ref-73)
74. See *Méthode de l’Universalité*, chap. VI : “Cavalieri, Mr Fermat, Mr Wallis, et autres supposent des certaines lettres, ou lignes infinement petites ou egales a rien. J’ay mis la mesme chose en usage, et j’ay adjousté des lettres qui representent une grandeur infinie, ou des lignes egales à des rectangles, comme sont les asymptotes de l’Hyperbole“ (VII, 7, 79). [↑](#footnote-ref-74)
75. Leibniz sent a first text as a letter to Pinsson, which was published almost *in extenso* in the *Mémoire de Trevoux*. This text, a reaction to critiques raised by Father Gouye, raised a lot of perplexity, even amongst his supporters, since it contained a comparison between the various orders of differentials and fixed and finite entities such as a grain of sand and the sun (GM V 96). Being asked about these comparisons by Varignon, Leibniz replied in the famous letter from Feb 2 1702, that it was a coarse way of speaking and that infinitesimals should not be seen as fixed entities. This letter was then published in the *Journal des Savants* (*Extrait d’une lettre à M. Varignon, contenant l’explication de ce qu’on a raporté de luy dans les Memoires de Trevoux des mois de Novembre et Decembre derniers*). The *Justification* was conceived as a follow up to this letter aimed specifically at critique coming from Rolle. It was send to Pinsson for Varignon for publication in the *Journal des Savants*, but the project did not succeed. [↑](#footnote-ref-75)
76. This is the interpretation in terms of “incomparables” to which we will come back later. [↑](#footnote-ref-76)
77. We follow the translation given by Jesseph in his (2015, 201). [↑](#footnote-ref-77)
78. “*un moyen fort palpable de justifier nostre maniere de calculer par le calcul ordinaire d’Algebre*” [↑](#footnote-ref-78)
79. This parallel has led some scholars to read the *Defense* as a version of the *Justification*. This has had the unfortunate consequence of hiding another new strategy announced in the *Defense* (and absent from the *Justification*) in which Leibniz proposes an interpretation devoid of infinitesimals. We’ll give our hypothesis about the proper status of this text below. [↑](#footnote-ref-79)
80. In one draft Leibniz thought about the possibility of positing the Law of Continuity as an axiom (LH XXXV, 29, 8 foL 1-2). [↑](#footnote-ref-80)
81. Let us note, for now, that when doing so, Leibniz will be very explicit about the general context in which he takes the Law of Continuity to hold in mathematics mentioning the examples dating back from the Parisian stay: parallel lines seen as meeting at infinity or ellipses transforming into parabolas when one of the focus goes at infinity. [↑](#footnote-ref-81)
82. Thus in his 2015 paper, D. Jesseph distinguishes two strategies that might reconcile Leibniz[’s] requirements for rigorous demonstration”, a “syntactic” or “proof-theoretic” one, and a “semantic” or “model-theoretic” approach (2015, 196-7). The former is roughly the idea that “a symbol like ‘*dx*’ is simply a placeholder for a much more elaborate line of reasoning that makes reference only to finite differences of finite quantities”, while the latter is the idea that “use of infinitesimals would never lead from truth to falsehood”, which would require “something like the proof of a principle that adding infinitesimals to the standard geometry yields a model-theoretic conservative extension of standard geometry” (197). On that view the infinitesimal would be “something like a Hilbertian ideal element” (202). Similarly, Katz and Sherry identify two different approaches, one relying on the Method of Exhaustion, the second embracing infinitesimals as fictions in the sense of “modern formalist positions such as Hilbert’s and Robinson’s” (Katz and Sherry 2012, 1553). They refer this dichotomy to Bos (2012, 1551; 2013, 575), although Bos seems to have been more cautious on this issue. [↑](#footnote-ref-82)
83. “I call those magnitudes incomparable of which one multiplied by any finite number whatsoever cannot exceed the other, in the same manner that Euclid takes it in the fifth definition of the fifth book. (Leibniz to L’Hôpital, GM II 287–289) [↑](#footnote-ref-83)
84. Similarly, Karin and Mikhail Katz comment “Leibniz repeatedly asserted that his infinitesimals, when compared to other quantities, violate the Archimedean property, viz., Euclid’s Elements, V.4… This appears directly to contradict Ishiguro’s claim that Leibniz was working with an Archimedean continuum.” (Katz and Katz, 3-4). [↑](#footnote-ref-84)
85. Cf. Leibniz to Luigi Grandi, September 1713: “Interea infinite parva concipimus non ut nihila simpliciter et absolute, sed, ut *nihila respective…*” (GM IV 218). (More on this below.) [↑](#footnote-ref-85)
86. Leibniz makes this point to Grandi in the same letter (GM IV 219). [↑](#footnote-ref-86)
87. That this transfers to the use of the differential algorithm is immediate since we just have to express the relationship in the symbolism of this calculus: this is what Leibniz does in his letter to Bodenhausen from 1690 and in the *Compendium* of the *Quadratura* written at the same period. [↑](#footnote-ref-87)
88. And what remains to be done *for us* would then be to show that these justification are fully compatible with the syncategorematic view, a task to which we shall turn after the presentation of Leibniz’s justifications. [↑](#footnote-ref-88)
89. Lettre to J. Bernoulli, 29 July 2019 (quoted above p. ???). Same statement in the letter to Bodenhausen A III, 5, 149, or at the beginning of the *Defense du calcul* « On leur peut tousjour monstrer que tout ce qui se conclut par ce calcul peut estre prouvé par une reduction *ad absurdum* à la façon d’Archimede: et en se servant des Lemmes des incomparables proposées dans les Actes de Leipzic. » (see below p. ??? for a translation). See also, in addition to the letter to Wallis of 1699, the *Responsio* GM V, 322. [↑](#footnote-ref-89)
90. For a general presentation, see Jesseph 1998. [↑](#footnote-ref-90)
91. GM IV, 50. [↑](#footnote-ref-91)
92. This argument plays a crucial role in his idea that Leibniz was fluctuating in his acceptance of infinitesimals as eliminable or not (Jesseph 2008, 215). [↑](#footnote-ref-92)
93. We saw in our study of Prop. 8 that this is not a trivial matter. [↑](#footnote-ref-93)
94. See (Bos 1974-5) and (Arthur 2013) for details. [↑](#footnote-ref-94)
95. Leibniz uses the comma to play a role similar to our parentheses. Here it indicates that the whole expression *xx + 2xdx + dxdx* has to be divided by *a*, and not just the last term. [↑](#footnote-ref-95)
96. Hence it does not seem possible to claim that “The assignable quantity (d)x passes via infinitesimal dx on its way to absolute 0” (Katz and Sherry 2013, 581). [↑](#footnote-ref-96)